Stability of Ideal Thyristor and Diode Switching Circuits

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Abstract—**This paper analyzes the stability of a general RLC circuit with ideal thyristors or diodes and periodic sources. Applications include high power thyristor controlled reactor and bridge rectifier circuits. The periodic steady states of the circuit are analyzed using a Poincaré map and transversality conditions are given to guarantee the smoothness of the Poincaré map. A simple and exact formula for the Jacobian of the Poincaré map is proved. Account is taken of the varying state space dimension as diodes switch on and off. When the transversality conditions fail, switching times can jump or bifurcate. Examples show that these switching time bifurcations can cause instability of thyristor circuits and mode changes of diode circuits. The simplification of the Jacobian formula is used to explain why the switching time bifurcations occur and are not predicted by the eigenvalues of the Jacobian. Periodic orbits of ideal diode circuits are proved to be stable using Jacobian and incremental energy methods. A source of damping in switching circuits is identified.**

I. Introduction

An ideal diode turns off and becomes an open circuit when its current decreases through zero and turns on and becomes a short circuit when its voltage increases through zero. An ideal thyristor is the same as an ideal diode except that its turn on is inhibited unless the thyristor firing pulse is on. This idealization is particularly useful and appropriate when analyzing the overall system performance of high power switching circuits attached to utility power lines.

Utility applications for thyristor switching circuits include rectifying and inverting bridge circuits for high voltage dc transmission, thyristor controlled reactors for static var control, and the emerging technology of flexible ac transmission [9], [11], [19]. Detailed systems studies of the effect of these devices on the power system use network models consisting of linear RLC elements, ideal thyristors, and periodic voltage, or current sources. The switching devices are controlled by varying the thyristor firing times. Even in the case of periodic thyristor firing (firing delay fixed with respect to the sources), little is established about the stability of these models. For example, the damping effect associated with the thyristors switching off and the instabilities due to switching time bifurcations [14]–[16] are poorly known. We analyze the stability of these systems in the case of periodic thyristor firing. It is convenient to first derive results for ideal diode circuits and then adapt the results to the ideal thyristor circuits since the ideal diodes have a cleaner theory. Ideal diodes and thyristors are also useful idealizations for some system studies at lower power levels [21], [22], [29], [32].

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We first consider a circuit of linear RLC elements, ideal diodes, periodic voltage, and current sources. These nonlinear circuits are usually operated in a periodic fashion so that steady state operation corresponds to a periodic orbit in state space. We exploit the special structure of these circuits to derive a simple formula for the stability of the periodic orbit. The formula is used to obtain conditions for the stability of the periodic orbit and to analyze the switching time bifurcations in which the switching times jump as circuit parameters are slowly varied.

Since the circuit is linear except for the diodes, the circuit is linear between diode switchings and may be analyzed as a succession of linear circuits. Each linear circuit depends on which diodes are conducting; the ideal diode model implies that on diodes are a short circuit and off diodes are open circuits. The initial state of the circuit affects the time of the diode switchings and hence the time at which the linear systems change. This dependence of the switching times on the state causes the nonlinearity of the circuit and is responsible for the difficulty and interest of the analysis. The dimension of the state space changes when the diodes switch, and this is taken into account by a change of coordinates at each switching.

Steady-state circuit operation corresponds to a periodic orbit in state space, and a natural way to analyze the circuit operation is to sample the circuit state every period. The map that advances the state by one period is called the Poincaré map [8], [31], and periodic orbits of the circuit correspond to fixed points of the Poincaré map. Moreover, the stability of a periodic orbit can be determined (except in borderline cases) from the Jacobian of the Poincaré map evaluated at the fixed point [8], [31]. In fact, the eigenvalues of the Poincaré map are Floquet multipliers of the periodic orbit. See [15] for a detailed example of a Poincaré map applied to a switching circuit.

The analysis of diode circuits easily adapts to thyristor circuits if the thyristor firing pulses are periodic and the thyristors do not misfire so that they turn on periodically. Indeed, the analysis is simplified by the periodic thyristor turn on time. (The onset of thyristor misfire is described as a transcritical bifurcation in [26].)

The main objective of this paper is to give a comprehensive account of stability and bifurcations of a general circuit with RLC elements, periodic sources, and ideal thyristors or diodes. Sections I and II develop the circuit equations and switching conditions, and Sections III–VI derive a simple and exact formula for the Jacobian of the Poincaré map. One might expect the formula for the Jacobian to be complicated by the nonlinear dependence of the switching times on the circuit state, but we prove that the formula simplifies. The Jacobian formula is illustrated with a diode bridge rectifier circuit in Section VI, and a thyristor circuit example may be found in [14], [15]. A brief account of the Jacobian simplification appeared in [6].

The Jacobian of the Poincaré map is basic to studying stability of a periodic orbit of the circuit. Section VII constructs "nice" coordinates in which the Euclidean norm is related to the energy stored in the inductors and capacitors. Section VIII uses the nice coordinates and the Jacobian formula to show that the Jacobian eigenvalues are always inside the unit circle. If the circuit is dissipative, then the Jacobian eigenvalues are strictly inside the unit circle, and the corresponding periodic orbits are asymptotically stable. However, switching times can change discontinuously, and Section IX describes these switching time bifurcations as fold bifurcations of diode currents or voltages. Examples show that switching time bifurcations can cause instability of thyristor circuits and mode changes of diode circuits as described in Section X. The switching time bifurcations are not predictable from the eigenvalues of the Jacobian of the Poincaré map and can occur when the eigenvalues are strictly inside the unit circle. Section XI uses the Jacobian simplification to explain this novel phenomenon. Section XII proves that periodic orbits of diode circuits are stable by using incremental energy methods [29].

In deriving the Jacobian of the Poincaré map, we use a state space analysis of switching circuits that overlaps with contributions of other authors. The fundamental work of Louis [21] computes Poincaré maps for switching circuits including controls. The varying dimensions of the state vector and switching conditions are discussed, and formulas for the propagation of first order deviations through switchings are stated. Louis computes as an example the Jacobian of the Poincaré map of an ac/dc convertor with a current regulator. Many of the ideas in Sections II–VI of this paper can be found in [21], but the development and derivations differ greatly. In particular, our development constructs the coordinate changes systematically, includes regularity conditions on the switchings, and gives a detailed proof of the Jacobian simplification. However, our paper does not take account of controls as [21] does.

Other authors have also used a state space approach to compute Jacobians for switching circuits. Verghese *et al.* [32] give a general approach to computing Poincaré maps and their Jacobians for switching circuits. Circuit controls and symmetries and the automation of the computations are discussed, but the Jacobian simplification and the varying dimension of the state vector are not treated. The linearized dynamics of a series resonant convertor are computed and studied. Grotzbach and Lutz [7] compute Jacobians of Poincaré maps of switched circuits including control actions. They develop Newton algorithms for computing steady state solutions and compute eigenvalues for ac/dc convertors with controls. The extension to nonlinear circuits and the derivation of averaged circuit models are discussed. Bedrosian and Vlach [2] extend the approach in [1] to give formulas for the Jacobian of a general switching circuit and compute the steady state of convertor circuits using Newton's method. The Jacobian formula applicable to ideal diode switchings in [2] is from [20] and can only be generally valid when the minimum state space dimension encountered as the circuit evolves is one or zero because the vector outer product in the Jacobian formula implies that the Jacobian has maximum rank one.

Chua *et al.* [3] prove an analogous Jacobian simplification for a nonlinear resonant circuit with a sinusoidal voltage source, piecewise linear inductor, and a linear capacitor and resistors. They analyze this circuit as a succession of linear circuits of the *same* dimension, which vary continuously across the switchings and derive a simplified formula for the Poincaré map Jacobian. The formula is used to compute the stability of periodic orbits, and subharmonic and chaotic circuit solutions are investigated. Shaw and Holmes [30] compute a simple Jacobian for a mechanical piecewise linear oscillator. Parker and Chua [25] and Hasler [10] prove that the simplified Jacobian is correct for a general piecewise linear circuit. The ideal diode or thyristor model differs from the piecewise linear assumption because the ideal diodes or thyristors cause the state space dimension to change at each switching. Moreover, the inhibition of thyristor turn on when the firing pulse is off has no counterpart in a piecewise linear model. Inaba and Mori [12], [13] study circuits related to the forced Van der Pol and Rayleigh oscillators, which contain a diode and a negative conductance by letting a piecewise linear diode model tend to an ideal diode model with a forward voltage drop. A one dimensional Poincaré map and its Jacobian are analytically derived and used to study toroidal and chaotic solutions.

Many special purpose computer codes to simulate rectifiers and other switching circuits have been written, and these can be used to study transients and stability on a case by case basis. The computer codes that most closely reflect the analytic approach of this paper are those which numerically integrate the circuit equations until a diode switching is detected, reformulate the circuit equations with the off diode branches removed and the on diode branches inserted and then continue integrating the reformulated circuit equations until the next switching [1], [4], [23], [24].

Jalali *et al.* [14], [15] present simulation and experimental evidence for switching time bifurcations in a basic thyristor controlled reactor circuit for static VAR control. This paper complements and generalizes [15] by presenting a rigorous account of the theory of switching time bifurcations in diode and periodically fired thyristor circuits. We suggest reading [15] before this paper.

II. System Equations with a Diode On or Off

We construct general system equations for a circuit with a particular diode on or off. The state space changes when the diode switches, and we consider the coordinate change relating the state space when the diode is on to the state space when the diode is off.

When the diode is on, the circuit dynamics are described by the linear system

$$\dot{x} = Ax + Bu \tag{2.1}$$

where $x(t) \in \mathbf{R}^n$ is the system state vector and $u(t) \in \mathbf{R}^m$ is a vector of smooth functions of time representing the sources. (Throughout the paper, "smooth" may be read as "C^∞" or "C^2.") A and B are constant matrices, and x and u are written as column vectors. The state vector is assumed to be continuous across the switchings. This holds, for example, when the states are inductor currents and capacitor voltages.

We assume throughout the paper

Assumption 2.1: Each diode has a cutset κ consisting only of inductors (at least one) and current sources.

For example, Assumption 2.1 holds when the diode has an inductance in series and does not hold if the diode has a capacitor connected across its terminals in parallel. Another assumption that could be made in place of Assumption 2.1 would require each diode to have a loop consisting only of capacitors (at least one) and voltage sources. Results similar to those of this paper could be obtained for diode circuits with this alternative assumption, but we neglect this possibility because Assumption 2.1 is more useful for high power circuits.

The diode current i_d is a linear function of the state x and the input u specified by the row vectors c and b so that

$$i_d = cx + bu. \tag{2.2}$$

Assumption 2.1 implies that the diode cutset κ contains some inductors and hence that $c \neq 0$. When the diode is off, its current is zero, and $cx = -bu$; that is, the circuit state is constrained to lie in a hyperplane with normal c. In many circuits, $b = 0$ and the hyperplane passes through the origin and is fixed. In general, $b \neq 0$, and the hyperplane moves along its normal as u varies with time but maintains its orientation normal to c. The state of the circuit with the diode off can be represented by a vector y in the hyperplane. The hyperplane and y are $(n-1)$ dimensional since one degree of freedom of the circuit is lost when the diode switches off.

Choose the columns of an $n \times (n-1)$ matrix Q as a basis for vectors in the hyperplane. Since the columns of Q are independent, Q must have rank $n-1$. Q is called an injection matrix. Then y (hyperplane coordinates) is related to x (original coordinates for $\mathbf{R}^n$) by

$$x = Qy - Zu \tag{2.3}$$

where the matrix Z is chosen so that

$$cZ = b. \tag{2.4}$$

Also, since c is normal to any vector in the hyperplane,

$$cQ = 0. \tag{2.5}$$

When the diode is off, it is convenient to replace the off diode by a voltage source $-v_d$. Then

$$\dot{x} = Ax + Bu - d'v_d \tag{2.6}$$

for some constant column vector d'. Since it is shown at the end of the Section III that $cd' > 0$, we can define a scaled version of d' as $d = d'/(cd')$ so that

$$cd = 1. \tag{2.7}$$

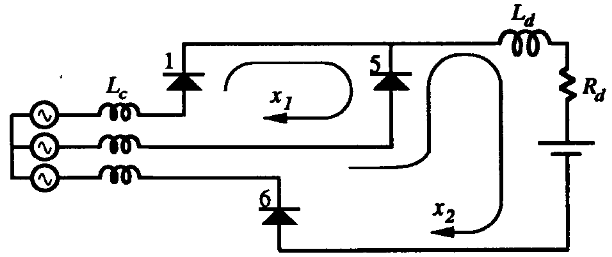

Fig. 1. Commutating diode bridge circuit.

That is, d is a nonzero vector not in the hyperplane. It follows that the matrix $(Q \mid d)$ is invertible, and we define the first $n-1$ rows of the inverse to be the $(n-1) \times n$ matrix P

$$(Q|d)^{-1} = \begin{pmatrix} P \\ \hline c \end{pmatrix}. \tag{2.8}$$

The last row of the inverse is c because $cQ = 0$ and $cd = 1$. It follows from

$$\begin{pmatrix} P \\ \hline c \end{pmatrix}(Q|d) = \begin{pmatrix} PQ & Pd \\ cQ & cd \end{pmatrix} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 1 \end{pmatrix} = I_n$$

that

$$PQ = I_{n-1} \tag{2.9}$$

and

$$Pd = 0. \tag{2.10}$$

Also

$$(Q|d)\begin{pmatrix} P \\ \hline c \end{pmatrix} = QP + dc = I_n. \tag{2.11}$$

Equations (2.9) and (2.10) imply that P has rank $n-1$, and the kernel of P is $\langle d \rangle$. P can be geometrically interpreted as the projection onto the hyperplane along d.

We choose the matrix Z to be

$$Z = db \tag{2.12}$$

(recall that d is a column vector and b is a row vector). Then $cZ = cdb = b$ so that the previous requirement (2.4) on Z is satisfied. Moreover, (2.10) implies that

$$PZ = 0. \tag{2.13}$$

Equation (2.3) becomes

$$x = Qy - dbu. \tag{2.14}$$

Multiplying both sides of (2.14) by P and using (2.10) yields

$$y = Px. \tag{2.15}$$

Multiplying the off diode equations (2.6) by P and using (2.15) and (2.14) allows the off diode equations to be written in the y coordinates as the linear system

$$\dot{y} = PAQy + PBu - PAdbu. \tag{2.16}$$

Consider the circuit of Fig. 1 as an example. When all three diodes conduct, the state space is two-dimensional, and the state $x = (x_1, x_2)^t$ consists of the two currents x_1, x_2 shown in Fig. 1. The circuit equations are

$$\dot{x} = A_{\text{on}}x + B_{\text{on}}u$$

$$\text{where} \quad A_{\text{on}} = \frac{-R_d}{(3L_c + 2L_d)}\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}. \tag{2.17}$$

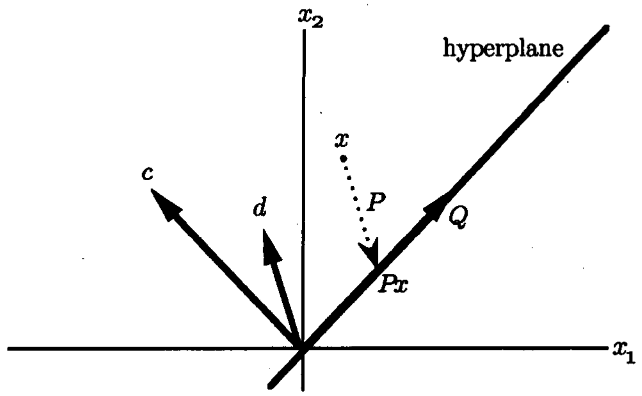

Fig. 2. State space geometry.

The current i_d in diode 5 is $i_d = x_2 - x_1 = cx$ where $c = (-1, 1)$. In this case, $b = 0$. When diode 5 turns off, the state is restricted to the hyperplane (in this case a line) $0 = x_2 - x_1 = cx$. c is normal to this hyperplane as shown in Fig. 2. $Q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a basis for vectors in the hyperplane, and the hyperplane coordinate y is related to x by $x = Qy$. Write $L = 2L_c + L_d$. Then $d' = \frac{1}{L_c(2L - L_c)} \begin{pmatrix} -L_c - L_d \\ L_c \end{pmatrix}$ specifies how the off diode 5 voltage contributes to (2.6) and $d = \frac{1}{L} \begin{pmatrix} -L_c - L_d \\ L_c \end{pmatrix}$. Compute $P = (L_c/L, (L_c + L_d)/L)$ from (2.8). P can be interpreted as a projection onto the hyperplane as indicated by the dotted line parallel to d in Fig. 1. The circuit equations when diode 5 is off are

$$\dot{y} = A_{\text{off}} y + P B_{\text{on}} u \quad \text{where} \quad A_{\text{off}} = P A_{\text{on}} Q = -R_d/L. \tag{2.18}$$

III. Diode Switching Conditions

The switching conditions on the diode current and voltage are formulated and related to the coordinate changes specified by the P and Q matrices. Transversality conditions usually satisfied at switchings are stated.

When the diode is on, the diode current $i_d = cx + bu$ and the diode switch off condition is

$$cx + bu = 0. \tag{3.1}$$

When the diode is off, $c\dot{x} = -b\dot{u}$ so that multiplying (2.6) by c yields

$$v_d = (cd')^{-1}[c(Ax + Bu) + b\dot{u}]$$

and the diode switch on condition

$$c(Ax + Bu) + b\dot{u} = 0. \tag{3.2}$$

Rearranging (2.11) yields the important identity

$$I_n - QP = dc \tag{3.3}$$

which relates the coordinate changes P and Q to the vector c determining the diode switching conditions.

At a typical switch off of an ideal diode, the diode current i_d encounters zero with a negative gradient. More precisely, if the diode current is regarded as a function of time, then its time derivative $\frac{di_d}{dt}$ satisfies

$$\frac{di_d}{dt}(s_{\text{off}}-) < 0 \tag{3.4}$$

at the switching time s_{off}. Similarly, at a typical switch on, the diode voltage v_d encounters zero with a positive gradient

$$\frac{dv_d}{dt}(s_{\text{on}}-) > 0. \tag{3.5}$$

Now we show that $cd' > 0$ by assuming that the transversality condition (3.4) can be satisfied for at least one switch off of the diode. That is, suppose that there is an input u and a circuit initial state such that at a diode turn off at time s_{off}, the diode current i_d satisfies (3.4). This is a very mild assumption. The on system equations (2.1) and (2.2) imply that

$$\begin{aligned} \frac{di_d}{dt}(s_{\text{off}}-) &= c\dot{x}(s_{\text{off}}-) + b\dot{u}(s_{\text{off}}) \\ &= c[Ax(s_{\text{off}}) + Bu(s_{\text{off}})] + b\dot{u}(s_{\text{off}}). \end{aligned} \tag{3.6}$$

Since the state is continuous across the switching, and the input is smooth, $x(s_{\text{off}})$, $u(s_{\text{off}})$ and $\dot{u}(s_{\text{off}})$ are not ambiguous. After the switch off, $0 = i_d = \frac{di_d}{dt}$. Then the off system equations (2.6) and (2.2) imply that

$$\begin{aligned} 0 = \frac{di_d}{dt} &= c\dot{x}(t) + b\dot{u}(t) \\ &= c[Ax(t) + Bu(t)] - cd'v_d + b\dot{u}(t). \end{aligned} \tag{3.7}$$

Evaluate (3.7) at $s_{\text{off}}+$ and use (3.6) to obtain

$$cd'v_d(s_{\text{off}}+) = \frac{di_d}{dt}(s_{\text{off}}-). \tag{3.8}$$

Now (3.8) and (3.4) imply that $cd' > 0$. Equation (3.8) states that the inductive voltage associated with the slope of the decreasing current just before the switching appears just after the switching as a negative voltage across the diode. Also, differentiating the on system equations and (3.7) once more similarly yields

$$cd'\dot{v}_d(s_{\text{off}}+) = \frac{d^2 i_d}{dt^2}(s_{\text{off}}-). \tag{3.9}$$

IV. Analysis of an Interval Containing a Diode Switch Off

Sections IV–VI derive a simple formula for the Jacobian of a periodic orbit. The approach is to divide one period of operation into subintervals, each of which contains one diode switching. Sections IV and V compute the Jacobian of the map, which advances the state from the beginning to the end of each subinterval. Then the chain rule is used in Section VI to compute the Jacobian of the Poincaré map as the product of the Jacobians for the subintervals.

Let $[t_1, t_2]$ be a time interval including a single diode switch off at time s_{off} and no other switchings. The switch off is assumed to satisfy the transversality condition (3.4). We write ϕ_1 for the flow that maps the state at t_1 to the state at t_2 so that

$$y(t_2) = \phi_1(x(t_1), t_1, t_2).$$

This section computes ϕ_1 and its Jacobian $D\phi_1$ with respect to $x(t_1)$. The proof that ϕ_1 is smooth and hence differentiable is postponed to the end of the section.

The diode is on in $[t_1, s_{\text{off}}]$ so that integrating (2.1) yields

$$x(s_{\text{off}}) = e^{A(s_{\text{off}}-t_1)}\left(x(t_1) + \int_{t_1}^{s_{\text{off}}} e^{A(t_1-\tau)}Bu(\tau)\,d\tau\right). \tag{4.1}$$

The transformation to y coordinates at the switch off time s_{off} is

$$y(s_{\text{off}}) = Px(s_{\text{off}}). \tag{4.2}$$

The diode is off in $[s_{\text{off}}, t_2]$ so that integrating (2.16) with initial condition $y(s_{\text{off}})$ gives

$$\begin{aligned} y(t_2) &= \phi_1(x(t_1), t_1, t_2) \\ &= e^{PAQ(t_2-s_{\text{off}})}\, y(s_{\text{off}}) \\ &\quad + \int_{s_{\text{off}}}^{t_2} e^{PAQ(t_2-\tau)}P(B - Adb)u(\tau)d\tau. \end{aligned} \tag{4.3}$$

Substituting for $y(s_{\text{off}})$, $x(s_{\text{off}})$ from (4.2), (4.1) yields

$$\begin{aligned} \phi_1(x(t_1), t_1, t_2) &= e^{PAQ(t_2-s_{\text{off}})}Pe^{A(s_{\text{off}}-t_1)} \\ &\quad \times \left(x(t_1) + \int_{t_1}^{s_{\text{off}}} e^{A(t_1-\tau)}Bu(\tau)\,d\tau\right) \\ &\quad + \int_{s_{\text{off}}}^{t_2} e^{PAQ(t_2-\tau)}P(B - Adb)u(\tau)\,d\tau. \end{aligned} \tag{4.4}$$

Differentiate with respect to $x(t_1)$ to obtain

$$\begin{aligned} D\phi_1 &= e^{PAQ(t_2-s_{\text{off}})}Pe^{A(s_{\text{off}}-t_1)} \\ &\quad + e^{PAQ(t_2-s_{\text{off}})}PA(I - QP) \\ &\quad \times \left[e^{A(s_{\text{off}}-t_1)}\left(x(t_1) + \int_{t_1}^{s_{\text{off}}} e^{A(t_1-\tau)}Bu(\tau)d\tau\right)\right. \\ &\quad \left. + dbu(s_{\text{off}})\right]Ds_{\text{off}}. \end{aligned} \tag{4.5}$$

Note that some of the terms associated with s_{off} in the limits of the two integrals of (4.4) cancel. The row vector Ds_{off} is the gradient of s_{off} with respect to $x(t_1)$. Substitute from (3.3), (4.1) and use (2.4) to obtain

$$\begin{aligned} D\phi_1 &= e^{PAQ(t_2-s_{\text{off}})}Pe^{A(s_{\text{off}}-t_1)} \\ &\quad + e^{PAQ(t_2-s_{\text{off}})}PAd\big[cx(s_{\text{off}}) + bu(s_{\text{off}})\big]Ds_{\text{off}}. \end{aligned} \tag{4.6}$$

The switching condition (3.1) determining s_{off} is $0 = cx(s_{\text{off}}) + bu(s_{\text{off}})$ which implies that the middle portion of the second term vanishes, and we obtain the surprising and simple result

$$D\phi_1 = e^{PAQ(t_2-s_{\text{off}})}Pe^{A(s_{\text{off}}-t_1)}. \tag{4.7}$$

This simplification can be viewed another way. For fixed initial time t_1, the right hand side of (4.4) may be regarded as a function $\psi_1(x(t_1), s_{\text{off}})$ depending explicitly on the switching time s_{off}. Inspection of the right hand side of (4.4) shows that ψ_1 is a smooth function. In this formulation, the system equations are

$$y(t_2) = \psi_1(x(t_1), s_{\text{off}}) \tag{4.8.1}$$

$$0 = f(s_{\text{off}}, x(t_1)) \tag{4.8.2}$$

where

$$\begin{aligned} f(t, x(t_1)) &= cx(t) + bu(t) \\ &= ce^{A(t-t_1)}\left(x(t_1) + \int_{t_1}^{t} e^{A(t_1-\tau)}Bu(\tau)d\tau\right) \\ &\quad + bu(t) \end{aligned} \tag{4.9}$$

is the diode current assuming that the diode does not turn off. The zero diode current equation (4.8.2) is regarded as a constraint to be solved to determine s_{off} as a function of $x(t_1)$. Then it is well known (e.g. [32]) that differentiating $\phi_1(x(t_1)) = \psi_1(x(t_1), s_{\text{off}})$ yields

$$D\phi_1 = D\psi_1 + \frac{\partial \psi_1}{\partial s_{\text{off}}}Ds_{\text{off}} \tag{4.10}$$

where $D\psi_1$ is the first term of (4.5) and $\frac{\partial \psi_1}{\partial s_{\text{off}}}Ds_{\text{off}}$ is the second term of (4.5). However, the simplification of (4.6) shows that

$$\frac{\partial \psi_1}{s_{\text{off}}} = e^{PAQ(t_2-s_{\text{off}})}PAd[cx(s_{\text{off}}) + bu(s_{\text{off}})] = 0. \tag{4.11}$$

That is, the final state $y(t_2)$ is independent of the switching time to first order.

It remains to prove that the transversality condition (3.4) at the switch off implies that ϕ_1 is a smooth function. Since the diode current $f(t, x(t_1))$ defined in (4.9) is a smooth function, Lemma A1 of Appendix A proves that s_{off} is a smooth function of the initial state $x(t_1)$. Since $\phi_1(x(t_1)) = \psi_1(x(t_1), s_{\text{off}})$ and ψ_1 is a smooth function, it follows that ϕ_1 is smooth.

V. Analysis of an Interval Containing a Diode Switch On

Let $[t_2, t_3]$ be a time interval including a diode switch on at time s_{on} and no other switchings. We write ϕ_2 for the flow that maps the state at t_2 to the state at t_3 so that

$$x(t_3) = \phi_2(y(t_2), t_2, t_3). \tag{5.1}$$

This section computes ϕ_2 and its Jacobian $D\phi_2$ with respect to $y(t_2)$. We assume the transversality condition (3.5) at the switch on to ensure that ϕ_2 is a smooth function.

The diode is off in $[t_2, s_{\text{on}}]$ so that integrating (2.16) yields

$$\begin{aligned} y(s_{\text{on}}) &= e^{PAQ(s_{\text{on}}-t_2)} \\ &\quad \times \left(y(t_2) + \int_{t_2}^{s_{\text{on}}} e^{PAQ(t_2-\tau)}P(B - Adb)u(\tau)d\tau\right). \end{aligned} \tag{5.2}$$

The equation transforming to the x coordinates at s_{on} is

$$x(s_{\text{on}}) = Qy(s_{\text{on}}) - dbu(s_{\text{on}}). \tag{5.3}$$

The diode is on in $[s_{\text{on}}, t_3]$ so that integrating (2.1) and using (5.3) yields

$$\begin{aligned} \phi_2(y(t_2), t_2, t_3) &= e^{A(t_3-s_{\text{on}})}(Qy(s_{\text{on}}) - dbu(s_{\text{on}})) \\ &\quad + \int_{s_{\text{on}}}^{t_3} e^{A(t_3-\tau)}Bu(\tau)\,d\tau. \end{aligned} \tag{5.4}$$

Substitute from (5.2) and differentiate with respect to $y(t_2)$ to obtain

$$\begin{aligned} D\phi_2 &= e^{A(t_3-s_{\text{on}})}Qe^{PAQ(s_{\text{on}}-t_2)} \\ &- e^{A(t_3-s_{\text{on}})}(I-QP)AQe^{PAQ(s_{\text{on}}-t_2)}\Big(y(t_2) \\ &+ \int_{t_2}^{s_{\text{on}}} e^{PAQ(t_2-\tau)}P(B-Adb)u(\tau)\,d\tau\Big)Ds_{\text{on}} \\ &- e^{A(t_3-s_{\text{on}})}(I-QP)(B-Adb)u(s_{\text{on}})Ds_{\text{on}} \\ &- e^{A(t_3-s_{\text{on}})}db\dot{u}(s_{\text{on}})Ds_{\text{on}}. \end{aligned} \tag{5.5}$$

Substitute from (3.3), (5.2), and (2.12) to obtain

$$\begin{aligned} D\phi_2 &= e^{A(t_3-s_{\text{on}})}Qe^{PAQ(s_{\text{on}}-t_2)} - e^{A(t_3-s_{\text{on}})}d \\ &\times [c(AQy(s_{\text{on}}) - Adbu(s_{\text{on}}) + Bu(s_{\text{on}})) \\ &+ b\dot{u}(s_{\text{on}})]Ds_{\text{on}}. \end{aligned} \tag{5.6}$$

But (2.14) implies that the expression in square brackets is $[c(Ax(s_{\text{on}}) + Bu(s_{\text{on}})) + b\dot{u}(s_{\text{on}})]$, and this vanishes according to the switching condition (3.2) determining s_{on} so that we obtain the simple result

$$D\phi_2 = e^{A(t_3-s_{\text{on}})}Qe^{PAQ(s_{\text{on}}-t_2)}. \tag{5.7}$$

VI. Jacobian of a Periodic Orbit

Suppose the circuit is operating in a periodic fashion. That is, the circuit waveforms, sources, and the patterns of diode switching are periodic with period T. Then the circuit operation can be studied with the Poincaré map F_{t_1}, which maps the circuit state at time t_1 to the circuit state at time $t_1 + T$.

Conditions for the smoothness of the Poincaré map, which we assume in this section, are given by

Lemma 6.1: Suppose that each switch off in the period satisfies the transversality condition (3.4), and each switch on in the period satisfies the transversality condition (3.5). Then the Poincaré map F_{t_1} is smooth.

Lemma 6.1 is proved by dividing the period $[t_1, t_1 + T]$ into subintervals each containing one switching. According to Sections IV and V, the transversality conditions (3.4) and (3.5) imply that the maps advancing the state over each subinterval are smooth. Then F_{t_1} is smooth since F_{t_1} is the composition of the smooth maps over the subintervals.

A periodic orbit of the system passing through state x at time t_1 corresponds to a fixed point of the Poincaré map so that $F_{t_1}(x) = x$ and the stability of the periodic orbit may be determined (except in borderline cases) from the Jacobian $DF_{t_1}|_x$ of the Poincaré map evaluated at x.

Sections IV and V have computed Jacobians of the maps advancing time over subintervals of the period that contain one switching. According to the chain rule, the Jacobian $DF_{t_1}|_x$ can be computed by dividing the period into subintervals each containing one switching and multiplying the Jacobians of these subintervals. The general formula for $DF_{t_1}|_x$ stated at the end of the section is made apparent by first computing $DF_{t_1}|_x$ for the case of a period containing one diode switch on and one diode switch off.

Suppose that the time t_1 occurs when the diode is on, and we choose t_1 as the start of the period. Then the Poincaré map may be expressed as

$$F_{t_1}(x) = \phi_2(\phi_1(x, t_1, t_2), t_2, t_1 + T) \tag{6.1}$$

where t_2 is a time when the diode is on. Note that $t_1 \le s_{\text{off}} \le t_2 \le s_{\text{on}} \le t_1 + T$. It is straight forward to check using (4.4) and (5.4) that F_{t_1} is independent of the choice of t_2 in the interval $[s_{\text{off}}, s_{\text{on}}]$. The Jacobian is now easily obtained from (4.7), (5.7) and the chain rule

$$\begin{aligned} DF_{t_1}|_x &= D\phi_2|_{(\phi_1(x),t_2,t_1+T)}D\phi_1|_{(x,t_1,t_2)} \\ &= e^{A(t_1+T-s_{\text{on}})}Qe^{PAQ(s_{\text{on}}-s_{\text{off}})}Pe^{A(s_{\text{off}}-t_1)}. \end{aligned} \tag{6.2}$$

A slightly simpler form may be obtained by evaluating (6.2) at $t_1 = s_{\text{off}}$ and writing $\sigma = T - (s_{\text{on}} - s_{\text{off}})$ for the time interval when the diode is on

$$DF_{s_{\text{off}}}|_x = e^{A\sigma}Qe^{PAQ(T-\sigma)}P. \tag{6.3}$$

Alternatively, choose t_2 as the start of the period. Then the Poincaré map may be expressed as

$$F_{t_2}(y) = \phi_1(\phi_2(y, t_2, t_1), t_1, t_2 + T) \tag{6.4}$$

and the Jacobian can be similarly derived as

$$\begin{aligned} DF_{t_2}|_y &= D\phi_2|_{(\phi_1(y),t_1,t_2+T)}D\phi_1|_{(y,t_2,t_1)} \\ &= e^{PAQ(t_2+T-s_{\text{off}})}Pe^{A(s_{\text{off}}-s_{\text{on}})}Qe^{PAQ(s_{\text{on}}-t_2)}. \end{aligned} \tag{6.5}$$

The Jacobian $DF_{t_1}|_x$ of (6.2) is an $n \times n$ matrix whereas the Jacobian $DF_{t_2}|_y$ of (6.5) is an$(n-1) \times (n-1)$ matrix. $DF_{t_1}|_x$ has an additional zero eigenvalue not present in $DF_{t_2}|_y$ because of the rank $n-1$ matrices P and Q.

The switching times s_{off} and s_{on} in (6.5) depend on the state y and the circuit sources u. Thus, the Jacobian does depend on the state and the sources but only implicitly via the switching times s_{off} and s_{on}. This nicely shows the nature of the nonlinearity of the circuit. It is remarkable that the Jacobian formula (6.5) is the same simple formula that would be obtained by naively regarding the switching times as constants!

A slightly simpler form may be obtained by evaluating (6.5) at $t_2 = s_{\text{on}}$ and writing $\sigma = s_{\text{off}} - s_{\text{on}}$ for the time interval when the diode is on

$$DF_{s_{\text{on}}}|_y = e^{PAQ(T-\sigma)}Pe^{A\sigma}Q. \tag{6.6}$$

The action of (6.6) on a linearized perturbation δy may be informally described as: change to x coordinates with the injection Q, let the on system act for time σ, project to the off coordinates with P, and let the off system act for time$T - \sigma$.

The Jacobian formula for a general periodic orbit is now stated. Suppose the periodic orbit has k successive switching patterns and Poincaré map F. Switching state i has matrix A_i and lasts for time σ_i. Then

$$DF = e^{A_1\sigma_1}R_{12}e^{A_2\sigma_2}R_{23}\cdots e^{A_k\sigma_k}R_{k1} \tag{6.7}$$

where R_{ij} is the coordinate change relating the circuit state in switching pattern i to the circuit state in switching pattern

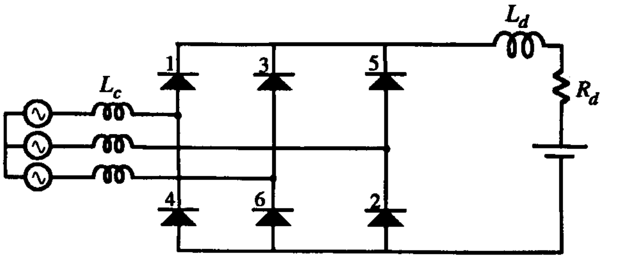

Fig. 3. Diode bridge circuit.

j. R_{ij} is a projection matrix (P_{ij}) if a diode switches off and an injection matrix (Q_{ij}) if a diode switches on. Since periodically fired thyristors have constant switch on times (we assume no misfires) and turn off just like a diode, (6.7) also holds for periodically fired thyristors.

We now derive as an example the Jacobian for the three phase diode bridge converter shown in Fig. 3. The dc load is a filter inductance L_d in series with a load resistance R_d and a constant voltage source. The ac system that feeds the diode bridge is modeled as a Thevenin ac source behind a commutating inductance L_c. The commutation time μ is assumed to be less than $T/6$.

Fig. 1 shows the commutating circuit starting from when diode 1 turns on to when diode 5 turns off. The equations for the commutating circuit and the circuit when diode 5 turns off and only diodes 1 and 6 conduct are presented at the end of Section II. Section II also computes the matrices A_{on} for the commutating circuit, $A_{\text{off}} = -R_d/L$ for the circuit just after diode 5 turns off, and $P = (L_c/L, (L_c + L_d)/L)$ relating the two circuits.

Just before diode 1 turns on, only diodes 5 and 6 conduct, and the circuit trajectories lie in a line (hyperplane) specified by $0 = x_1 = c_1 x$ where $c_1 = (1, 0)$. A basis for vectors in this line is $Q_1 = (0, 1)^t$. Note that the hyperplane specified by c_1, Q_1 when only diodes 5 and 6 conduct differs from the hyperplane specified by c, Q when only diodes 1 and 6 conduct.

Now consider the map ϕ which advances the state y_1 just before diode 1 turns on to the state y just before diode 2 turns on. Similarly to (6.6) the Jacobian of ϕ is

$$\begin{aligned} D\phi &= e^{A_{\text{off}}(T/6-\mu)} P e^{A_{\text{on}}\mu} Q_1 \\ &= e^{-(T/6-\mu)R_d/L} L^{-1} \\ &\quad \times (L_c, L_c + L_d) \begin{pmatrix} 1 & (e^{\alpha}-1)/2 \\ 0 & e^{\alpha} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

where $\alpha = -2R_d\mu/(3L_c + 2L_d)$. If we assume circuit symmetry, the Jacobian of the Poincaré map follows easily as $DF = [D\phi]^6$. In the special case of zero resistance, $R_d = 0$ and $DF = [D\phi]^6 = [P_5 Q_1]^6 = (L_c + L_d)^6/L^6$ which is independent of μ and agrees with [22]. See [14], [15] for another example.

VII. Nice Coordinates for a Switching Circuit

It is convenient to consider the switching of a particular diode and derive nice coordinates for the state both when the diode is on and when the diode is off. In this section, it does not matter whether the diode is turning on or off. Appendix B shows that nice coordinates x when the diode is on can be chosen so that $c = d^T$ and $|x|^2$ is the energy stored in the circuit inductances and capacitances. (Appendix B uses Assumption 2.1 and assumes that the inductance and capacitance matrices associated with the circuit differential equations are invertible.)

To construct compatible coordinates when the diode is off, choose the basis vectors for the hyperplane to be orthonormal so that the columns of Q are orthogonal and of unit length. Then since (2.5) and (2.7) yield $cQ = 0$ and $cd = cc^T = 1$, $(Q|d) = (Q|c^T)$ is orthogonal, and (2.8) implies that $P = Q^T$. Moreover, in the nice coordinates for the circuit with the diode off, $|y|^2 = y^T y = y^T Q^T Q y = x^T x = |x|^2$ so that $|y|^2$ is also the stored energy in the circuit inductors and capacitors. We call these norms "energy norms" for the state space when the diode is on or off.

Let $\|\cdot\|$ be the matrix norm induced by the energy norms. Then $\|Q\| = 1$ since $|Qy|^2 = y^T Q^T Q y = y^T y = |y|^2$ and $|Qy| = |y|$. Moreover, $\|P\| \le 1$ since (2.11) implies that $|x|^2 = x^T x = x^T(P^T P + c^T c)x = |Px|^2 + |cx|^2$ and $|Px| \le |x|$.

VIII. Local Stability

This section deduces the stability of a periodic orbit from the Jacobian formula (6.7) and the results of Section VII. All the diode switchings are assumed to satisfy the transversality conditions (3.4) or (3.5) so that Lemma 6.1 implies that the Poincaré map F is differentiable. The capacitance and inductance matrices for each circuit are assumed invertible so that the nice coordinates of Section VII may be chosen for the state space before and after each switching.

As in Section VI, suppose that the periodic orbit has k successive switching patterns and Poincaré map F. Switching pattern i has matrix A_i and lasts for time σ_i, and the coördinate change between successive switching patterns i and j is R_{ij}. Energy norms are chosen for the state space of each switching pattern. Consider the ith circuit with *zero input*. This linear RLC circuit has state transition matrix $e^{A_i t}$, and the energy stored in the inductors and capacitors cannot increase in this circuit. Therefore, in the induced matrix norm $\|e^{A_i t}\| \le 1$, $i = 1 \cdots k$. Moreover, the results $\|Q\| = 1$ and $\|P\| \le 1$ from Section VII imply that each $\|R_{ij}\| \le 1$. Applying these results to the Jacobian formula (6.7) yields

$$\|DF\| \le \|e^{A_1\sigma_1}\| \|R_{12}\| \|e^{A_2\sigma_2}\| \|R_{23}\| \cdots \|e^{A_k\sigma_k}\| \|R_{k1}\| \le 1$$

and hence that all the eigenvalues of DF lie within or on the unit circle. If the eigenvalues lie strictly inside the unit circle, then the periodic orbit is asymptotically stable. If some eigenvalues lie on the unit circle, then the stability of the periodic orbit is not determined by the Poincaré map linearization DF. In any case, the eigenvalues cannot leave the unit circle, and this precludes the periodic orbit becoming unstable via a conventional bifurcation. More precisely, the transversality conditions of generic bifurcations requiring the bifurcating eigenvalues to leave the unit circle at nonzero speed cannot be satisfied.

Asymptotic stability results are easy to obtain if the linear RLC circuits are also assumed to be dissipative so that the stored energy in the linear RLC circuits strictly decreases. For example, consider the periodic orbit of Section VI with a single diode switching on and off once and assume that the circuit with the diode on is dissipative so that $A < -\epsilon$ for some $\epsilon > 0$ (see Appendix B). The linear RLC circuit with the diode on and zero input has state transition matrix e^{At}. If $v(t)$ is any trajectory of this RLC circuit, then integration of $\frac{d}{dt}|v(t)| = \frac{d}{dt}\sqrt{v^T v} = v^T(A^T + A)v/(2|v|) \le -\epsilon|v|$ yields $|e^{At}v(0)| \le e^{-\epsilon t}|v(0)|$ so that $\|e^{At}\| \le e^{-\epsilon t}$. Moreover, since $A \le -\epsilon \Rightarrow PAQ = Q^TAQ \le -\epsilon$, it follows similarly that $\|e^{PAQt}\| \le e^{-\epsilon t}$. Now we obtain from the Jacobian formula (6.3)

$$\|DF_{s_{\rm off}}\| = \|e^{A\sigma}Qe^{PAQ(T-\sigma)}Px\| \le e^{-\epsilon\sigma}e^{-\epsilon(T-\sigma)} = e^{-\epsilon T}$$

so that the eigenvalues of $DF_{s_{\rm off}}$ lie strictly inside the unit circle within $e^{-\epsilon T}$ of the origin, and the periodic orbit is asymptotically stable.

We conclude that if the linear RLC circuits encountered during the periodic orbit are dissipative, then all periodic orbits that satisfy the transversality condition at each switching are asymptotically stable. However, in general, the periodic orbit may not be unique or globally stable; in the special case of periodically fired ideal thyristors in a basic static VAR control circuit, the Poincaré map is discontinuous where the transversality conditions are not satisfied, and the discontinuity allows two asymptotically stable periodic orbits in the state space [26].

IX. Switching Time Continuity and Bifurcations

This section describes how switching times vary smoothly with circuit parameters when transversality conditions on the circuit switchings are satisfied and how switching times can jump or bifurcate when the transversality conditions fail. The switching time bifurcations are fold bifurcations of the diode current or voltage.

In order to analyze stability using bifurcation theory, we introduce a real parameter λ that varies quasistatically and upon which the circuit elements or inputs smoothly depend. When this parameter dependence is taken into account, differential equations (2.1) for the circuit with the diode on become

$$\dot{x} = A(\lambda)x + B(\lambda)u(t,\lambda) \tag{9.1}$$

and the corresponding flow is written $\phi_{\rm on}(x(0), t, \lambda)$.

We describe only the bifurcation of a diode switch off because the analysis for a diode switch on is essentially the same. We suppose that at some particular time, which it is convenient to specify as time zero, the diode is on with positive current and that this situation persists over the parameter range of interest. That is, if we write $x_0(\lambda)$ for the state at time zero, then $i_d(0) = cx_0(\lambda) + bu(0,\lambda) > 0$. The initial state $x_0(\lambda)$ is assumed to be a smooth function of λ.

Define $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ to be the diode current assuming that the diode is on (short circuited) for all time

$$f(t,\lambda) = cx(t) = c\phi_{\rm on}(x_0(\lambda), t, \lambda) + bu(t,\lambda). \tag{9.2}$$

f is a smooth function of t and λ. Since the diode is assumed to be short-circuited, f can be negative and in practical periodically forced switching circuits, f will have many positive roots.

The first or smallest positive root of f is the diode switch off time $s_{\rm off}$

$$s_{\rm off}(\lambda) = \min\{s \mid f(s,\lambda) = 0 \quad \text{and} \quad s > 0\}. \tag{9.3}$$

Here we regard $s_{\rm off}(\lambda)$ as a function of λ. $f(t,\lambda)$ is identical to the actual diode current $i_d(t)$ for $t \in [0, s_{\rm off}]$. The first positive root of a function need not vary smoothly as parameters change because roots may disappear or new roots may be created. The switch off usually satisfies transversality condition (3.4), which may be rewritten in terms of f as

$$\frac{\partial f}{\partial t}(s_{\rm off}, \lambda_*) < 0 \tag{9.4}$$

and it follows that the switch off time varies smoothly as the parameter changes; Appendix A proves a slight generalization of

Lemma 9.1: If f is smooth, and the diode switch off at time $s_{\rm off}(\lambda_1)$ satisfies (9.4), then $s_{\rm off}(\lambda)$ is a smooth function of λ for λ sufficiently near λ_1.

It is easy to see that (9.4) implies via the implicit function theorem that there is a root $s(\lambda)$ of f with $s(\lambda_1) = s_{\rm off}(\lambda_1)$ that varies smoothly for λ near λ_1. However, the proof of Lemma 9.1 also establishes that no *new* roots are created in the interval $(0, s(\lambda))$ for λ sufficiently near λ_1. (Any new root in the interval $(0, s(\lambda))$ would prevent the root $s(\lambda)$ near $s_{\rm off}(\lambda_1)$ from being the *first* positive root of f required by definition (9.3).)

Jalali *et al.* [14], [15] describe in detail the discontinuous increase or decrease in switch off time as λ varies through a critical value λ_* as fold bifurcations of f in which pairs of roots of f disappear or appear. We briefly summarize this process before giving the precise conditions for the fold bifurcation. Consider the first three positive roots of f. Switching times discontinuously increase when the first and second roots of f coalesce to form a double root and then disappear in a fold bifurcation so that the switching time jumps forward to what was previously the third root. This process can be reversed to cause a switching time to discontinuously decrease. A new double root of f can be created by a fold bifurcation and then split into a first and second root in such a way that the previous switching time becomes the third root, and the switching time jumps backward to the new first root.

The first condition for the fold bifurcation is that f has zero gradient at the double root; that is, $\frac{\partial f}{\partial t}(s_{\rm off}, \lambda_*) = 0$ and (9.4) is not satisfied. The second condition $\frac{\partial f}{\partial \lambda}(s_{\rm off}, \lambda_*) \neq 0$ ensures that the bifurcation occurs as λ varies through λ_*, and the third condition $\frac{\partial^2 f}{\partial t^2}(s_{\rm off}, \lambda_*) > 0$ ensures the quadratic form of the graph of f at the double root. These transversality conditions are standard in bifurcation theory and expected to be generically satisfied (e.g. [8]). Fold bifurcations in the roots of a general smooth function satisfying these three conditions occur generically as a single parameter is varied. Since the function f associated with the diode current is not expected to be restricted by special symmetries or conditions, we expect switching time bifurcations to occur generically.

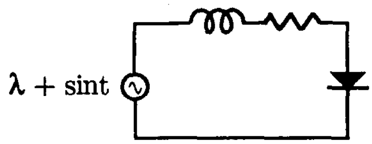

Fig. 4. Simple diode circuit.

X. EFFECTS OF SWITCHING TIME BIFURCATIONS

This section considers the effects of a switching time bifurcation on a stable periodic orbit of a circuit with thyristors or diodes. If a thyristor turn off in the periodic orbit undergoes a switching time bifurcation when the gate pulse is off, then the switch off time jumps to an earlier or later time. The typical consequence is that stability of the periodic orbit is lost, and a circuit transient starts. These switching time bifurcation instabilities have been observed in simulation and experimental work on a static VAR control circuit [14], [15].

If a thyristor turn off undergoes a switching time bifurcation when the gate pulse is off, then there is only one switching time created or destroyed because the gate pulse being off prevents the thyristor turning on just after the thyristor turn off. On the other hand, bifurcations of a diode switching off time can involve two switchings of the diode. That is, it is possible that both a switch off *and* a switch on are created or destroyed in the bifurcation. We suggest why this can typically happen. Consider a switch off of a diode at a switching time bifurcation at time $s_{\text{off}}(\lambda_*)$ and parameter value λ_*. The transversality conditions (3.4), (3.5) are assumed to be satisfied. At a turn off time $s_{\text{off}}(\lambda)$ just before the bifurcation destroys the switch off time, we have from (3.8) and (3.9)

$$v_d(s_{\text{off}}(\lambda)+) = \frac{1}{cd'}\frac{di_d}{dt}(s_{\text{off}}(\lambda)-) = \frac{1}{cd'}\frac{\partial f}{\partial t}(s_{\text{off}}(\lambda))$$
$$\dot{v}_d(s_{\text{off}}(\lambda)+) = \frac{1}{cd'}\frac{d^2 i_d}{dt^2}(s_{\text{off}}(\lambda)-) = \frac{1}{cd'}\frac{\partial^2 f}{\partial t^2}(s_{\text{off}}(\lambda).$$

Hence, as the bifurcation approaches, $v_d(s_{\text{off}}(\lambda)+) \to 0$, and $\dot{v}_d(s_{\text{off}}(\lambda)+) \to \frac{1}{cd'}\frac{\partial^2 f}{\partial t^2}(s_{\text{off}}(\lambda_*)) > 0$. Sufficiently close to the bifurcation, we have $v_d(s_{\text{off}}(\lambda)+) < 0$ and $\dot{v}_d(s_{\text{off}}(\lambda)+) > 0$. To first order

$$v_d\left(s_{\text{off}}(\lambda) + \frac{-v_d(s_{\text{off}}(\lambda))}{\dot{v}_d(s_{\text{off}}(\lambda))}\right)$$
$$= v_d(s_{\text{off}}(\lambda)) + \dot{v}_d(s_{\text{off}}(\lambda))\frac{-v_d(s_{\text{off}}(\lambda))}{\dot{v}_d(s_{\text{off}}(\lambda))} = 0.$$

That is, a linear prediction of the diode voltage predicts turn on at time

$$s_{\text{on}}(\lambda) = s_{\text{off}}(\lambda) + \frac{-v_d(s_{\text{off}}(\lambda))}{\dot{v}_d(s_{\text{off}}(\lambda))}.$$

$s_{\text{on}}(\lambda) > s_{\text{off}}(\lambda)$ and $s_{\text{on}}(\lambda) \to s_{\text{off}}(\lambda_*)$ as the bifurcation approaches. This suggests that a switch on quickly follows the switch off before it disappears and that the switch off and switch on coalesce together and disappear as the bifurcation occurs.

This behavior occurs in the simple diode circuit shown in Fig. 4 when the circuit resistance is positive. When the constant bias λ of the voltage source lies in the interval $(-1, 0)$, there is a unique and asymptotically stable periodic orbit in which the diode switches twice per cycle. In this mode, the Poincaré map at the periodic orbit with initial time when the diode is off simply maps zero current to zero current, and the Jacobian of the Poincaré map is zero. That is, a small perturbation in one cycle vanishes before the next cycle.

If λ increases through 0, the periodic orbit persists and remains asymptotically stable, but the diode never turns off. The stability of the periodic orbit is now governed by the resistor so that the Jacobian of the Poincaré map changes discontinuously when λ increases through zero. If λ decreases through -1, the periodic orbit becomes a constant zero current, and the diode never turns on. In both these switching time bifurcations, the two switching times coalesce and disappear. The effect of the switching time bifurcations is a mode change in the circuit, and asymptotic stability is not lost. Note, however, that in the extreme case of zero circuit resistance, the periodic orbit, although stable for $\lambda \leq 0$, disappears for $\lambda > 0$, and the circuit trajectory becomes unbounded.

In a general diode circuit, suppose $[t_1, t_2]$ is a time interval between two diode switchings corresponding to switching state j, and there is a switching time bifurcation at time $t_* \in [t_1, t_2]$ in which turn off and turn on of a diode appear together. The corresponding factor $e^{A_j(t_2-t_1)}$ of the Jacobian in (6.7) jumps to $e^{A_j(t_*-t_1)}QPe^{A_j(t_2-t_*)}$ as the bifurcation occurs. QP corresponds to a switch off and an immediately following switch on, where Q and P are the coordinate changes associated with the switching at t_*. This observation shows the discontinuity of the Jacobian of the Poincaré map at a switching time bifurcation.

XI. THE JACOBIAN DOES NOT PREDICT SWITCHING TIME BIFURCATIONS

Switching time bifurcations are not conventional bifurcations because they are not detectable or predictable from the eigenvalues of the Jacobian of the Poincaré map of the periodic orbit. For example, an asymptotically stable periodic orbit in a thyristor circuit with eigenvalues strictly inside the unit circle can encounter a switching time bifurcation and lose stability . This section uses the simplification of the Jacobian to explain this phenomenon.

We consider as an example the periodic orbit of Section VI, which contains one switch on and one switch off and suppose that the switch off undergoes a switching time bifurcation. The start of the period is denoted by t_2 and $t_2 < s_{\text{on}} < s_{\text{off}} < t_2 + T$.

The case of the switch off time s_{off} discontinuously decreasing by a fold bifurcation of f so that new roots of f appear in the interval $[s_{\text{on}}, s_{\text{off}}]$ is straightforward. The transversality condition continues to be satisfied at the root of f corresponding to the previous switch off time and the transversality condition is only violated at the new roots of f. Since the fold bifurcation occurs in the interior of the interval $[s_{\text{on}}, s_{\text{off}}]$, it is clear that the Jacobian and its eigenvalues are unrelated to the fold bifurcation.

The analysis of the Jacobian in the case of switching times suddenly increasing by the previous switch off time disappearing in a fold bifurcation is more subtle. Consider the formulation of the Poincaré map similarly to (4.8) as a map

G_{t_2} explicitly depending on s_{off} and a constraint equation to determine s_{off}

$$y(t_2+T) = G_{t_2}(y(t_2), s_{\text{off}}, \lambda) \tag{11.1}$$
$$0 = f(s_{\text{off}}, y(t_2), \lambda) \tag{11.2}$$

where $G_{t_2}(y,s,\lambda) = \psi_1(\phi_2(y,t_2,t_1),s)$ and $f(s,y,\lambda) = c\phi_2(y,t_2,s) = cx(s)$ is the diode current $i_d(t)$ for $t \le s_{\text{off}}$ and the diode current assuming that the diode does not turn off for $t \ge s_{\text{off}}$. The constraint (11.2) is simply $i_d(s_{\text{off}}) = 0$, and the transversality condition (3.4) becomes

$$\frac{\partial f}{\partial s}(s_{\text{off}}) < 0 \tag{11.3}$$

and the Jacobian DF_{t_2} is given by

$$DF_{t_2} = DG_{t_2} + \frac{\partial G_{t_2}}{\partial s} Ds_{\text{off}}. \tag{11.4}$$

If the switch off at s_{off} satisfies (11.3), then the gradient Ds_{off} of s_{off} with respect to $y(t_2)$ can be computed by differentiating and rearranging (11.2)

$$Ds_{\text{off}} = -\left(\frac{\partial f}{\partial s}\right)^{-1} Df. \tag{11.5}$$

As the fold bifurcation is approached, $\frac{\partial f}{\partial s}(s_{\text{off}}) \to 0$ and one might argue from (11.5) and (11.4) that $\frac{\partial f}{\partial s}(s_{\text{off}}) \to 0$ implies that $Ds_{\text{off}} \to \infty$ and that $\|DF_{t_2}\| \to \infty$. Moreover, $\|DF_{t_2}\| \to \infty$ implies that at least one eigenvalue of the Jacobian would leave the unit circle as the fold bifurcation is approached so that there would be a conventional bifurcation just before the switching time bifurcation is encountered. However, this argument does not apply because the simplification of the Jacobian in Section IV states that G_{t_2} is independent of the switch off time to first order so that $\frac{\partial G_{t_2}}{\partial s} = 0$ and hence that the Jacobian $DF_{t_2} = DG_{t_2}$ and is independent of the behavior of Ds_{off} (see (4.8–4.9)). That is, the eigenvalues of the Jacobian give no warning of the switching time bifurcation. It is argued above that if the simplification did not apply, then the circuit would typically lose stability in a conventional bifurcation before the switching time bifurcation was encountered. Thus, the simplification can be seen as essential in allowing the switching time bifurcation to occur.

XII. Damping of Incremental Energy in Ideal Diode Circuits

Sanders [29] uses incremental energy methods to prove that circuits with incrementally passive diode models are stable. This section extends these methods to prove the stability of an RLC circuit with ideal diodes and time dependent sources. Diodes are seen to damp incremental energy and contribute to circuit stability. The proof is more general than the results of Section VIII because it proves stability for any periodic orbit of the diode circuit even at a switching time bifurcation or when Jacobian eigenvalues are on the unit circle. However, the corresponding proof for thyristors fails, and we discuss this difference between diode and thyristor circuits.

Number all the circuit components, including on and off diodes. For a given trajectory of the circuit, let $i_k(t)$ and $v_k(t)$ be the current in and voltage across circuit component number k as a function of time t. For definiteness, current and voltage conventions are chosen such that an on diode conducts positive current and an off diode withstands negative voltage. Let $i'_k(t)$ and $v'_k(t)$ be the kth component current and voltages associated with a different circuit trajectory and define $\delta i_k = i_k(t) - i'_k(t)$ and $\delta v_k = v_k(t) - v'_k(t)$. Then, despite the circuit switchings, Tellegen's theorem applies to the incremental currents and voltages at every instant so that

$$\begin{aligned} 0 &= \sum_k \delta i_k \delta v_k \\ &= \sum_{\text{inductors}} \delta i_k \delta v_k + \sum_{\text{capacitors}} \delta i_k \delta v_k + \sum_{\text{resistors}} \delta i_k \delta v_k \\ &\quad + \sum_{\text{diodes}} \delta i_k \delta v_k + \sum_{\text{sources}} \delta i_k \delta v_k. \end{aligned} \tag{12.1}$$

Now use the nice state space coordinates x of Section VII to define the circuit incremental energy

$$|\delta x|^2 = \delta x^T \delta x \qquad \text{where} \quad \delta x = x(t) - x'(t).$$

By reexpressing the circuit incremental energy in terms of components [28]

$$|\delta x|^2 = \frac{1}{2}\sum_{\text{inductors}} L_k(\delta i_k)^2 + \frac{1}{2}\sum_{\text{capacitors}} C_k(\delta v_k)^2. \tag{12.2}$$

The incremental inductor currents δi_k and incremental capacitor voltages δv_k are smooth between switchings. Then for any time between switchings, differentiate (12.2) and use (12.1) to obtain

$$\begin{aligned} \frac{d}{dt}|\delta x|^2 &= \sum_{\text{inductors}} \delta i_k \delta v_k + \sum_{\text{capacitors}} \delta i_k \delta v_k \\ &= -\sum_{\text{resistors}} \delta i_k \delta v_k - \sum_{\text{diodes}} \delta i_k \delta v_k - \sum_{\text{sources}} \delta i_k \delta v_k. \end{aligned} \tag{12.3}$$

The summation over sources vanishes because $\delta v_k = 0$ for voltage sources and $\delta i_k = 0$ for current sources. The summation over resistors is greater than or equal to zero. Consider one of the terms of the summation over diodes

$$\delta i_k \delta v_k = (i_k(t) - i'_k(t))(v_k(t) - v'_k(t))$$
$$= \begin{cases} 0; & \text{diode is on for both trajectories} \\ 0; & \text{diode is off for both trajectories} \\ -i_k(t)\, v'_k(t); & \text{diode is on for undashed trajectory and off for dashed trajectory} \\ -i'_k(t)\, v_k(t); & \text{diode is off for undashed trajectory and on for dashed trajectory.} \end{cases}$$

Since the diode conducts positive current and withstands negative voltages, $\delta i_k \delta v_k \ge 0$ and

$$\sum_{\text{diodes}} \delta i_k \delta v_k \ge 0. \tag{12.4}$$

Hence, the right-hand side of (12.3) is less than or equal to zero and

$$\frac{d}{dt}|\delta x| \le 0$$

between switchings. Since $|\delta x|$ is continuous at switchings, integrate over a cycle to obtain

$$|x(T) - x'(T)| \leq |x(0) - x'(0)|$$

or

$$|F(x) - F(x')| \leq |x - x'|. \tag{12.5}$$

Equation (12.5) implies the stability of any fixed point of the Poincaré map F and the corresponding periodic orbit. Equation (12.5) also implies that F is continuous. If the inequality in (12.5) is strict for all $x \neq x'$ due to resistive losses, then uniqueness and global asymptotic stability of a periodic orbit can be deduced by Lyapunov methods [29].

The derivation of (12.4) shows that a trajectory perturbed from a periodic steady state experiences damping of its incremental energy related to the diode switchings during the time intervals when the switching state of the perturbed and steady state trajectories differ. In particular, this incremental energy analysis predicts damping associated with a diode turn off. This damping is also described by the zeroing of the incremental diode or thyristor current at switch off by the projection matrix P corresponding to the switch off in the formula for the Jacobian of the Poincaré map [5]. (Also, see the reduction in energy $|P\delta x| \leq |\delta x|$ due to applying P at the end of Section VII.) The incremental energy analysis also predicts a damping effect associated with a diode turning on, but the corresponding matrix Q in the Jacobian formula preserves energy ($|Q\delta x| = |\delta x|$). That is, the incremental energy analysis predicts damping while the Jacobian predicts no damping. This can be resolved by noting that when the diode turns on, it has zero voltage, and the diode current increases quadratically from zero. Hence, the diode current is zero to first order at turn on, and there is no damping effect to first order and no damping predicted by the Jacobian. Thus, the incremental energy damping at turn on is a second order effect. In contrast, at turn off, the diode voltage is immediately negative (see (3.8)), and this first order damping effect is apparent both from incremental energy considerations and the Jacobian.

The proof fails if thyristors are substituted for the diodes. In particular, (12.4) of the proof can fail because a thyristor does not turn on when its voltage becomes positive and the firing pulse is off. If two trajectories in a thyristor circuit are always close to each other, then since a thyristor turns off just like a diode, (12.4) will hold, and the incremental energy will be damped. (In the absence of a misfire, the thyristor will turn on at the same time for both trajectories.) However, if the two trajectories become far apart, either because of widely separated initial conditions or a nearby switching time bifurcation, then the positive thyristor voltage can make (12.4) fail, and the incremental energy between the trajectories can increase. Rajaraman *et al.* [26] give an example of a thyristor switching circuit with a discontinuous Poincaré map and two asymptotically stable periodic orbits.

XIII. Conclusion

We give a general account of the stability and bifurcations of RLC circuits with periodic sources and ideal thyristors or diodes. The main assumption is that each thyristor or diode has a cutset of inductors and current sources. The periodic steady states of the circuit are studied using the Poincaré map, and transversality conditions at the switchings are shown to guarantee the smoothness of the Poincaré map. A simple and exact formula for the Jacobian of the Poincaré map is proved, and the stability of the Jacobian is deduced by energy methods once special coordinates are chosen. It follows that stability cannot be lost by conventional bifurcations.

However, when the transversality conditions fail at a switching, the switching times typically jump in a switching time bifurcation. Switching off times are determined by the first positive root of the thyristor or diode current, and a switching off time bifurcates when the first positive root disappears or appears in a fold bifurcation. Switching time bifurcations of a thyristor circuit can cause the Poincaré map to be discontinuous and the periodic orbit to become unstable [14], [15]. The Poincaré map of diode circuits is continuous, and switching time bifurcations of a diode circuit can cause mode changes in the circuit operation and lack of differentiability of the Poincaré map. The simplification of the Jacobian formula is used to explain why the switching time bifurcations occur and are not predicted by the eigenvalues of the Jacobian.

Both incremental energy arguments and the Jacobian formula predict damping associated with diodes or thyristors turning off. This damping can contribute to stability of high power switching devices. In particular, Rajaraman *et al.* [27] report significant damping of subsynchronous power system oscillations by a thyristor controlled series capacitor and Jalali *et al.* report damping of a static VAR control circuit [5], [15].

The Jacobian of the Poincaré map describes the small signal stability of periodic orbits and is also useful in computing steady state solutions by Newton's method [2], [7], [21], [32]. The simplification of the Jacobian that follows from [21] and is newly derived in this paper shows that the Jacobian varies as a function only of the time spent in each switching pattern. The resulting simplified Jacobian formula is much more useful for computations and theory. The Jacobian typically becomes more complex when some of the switching times are controlled but the simplification of the part of the Jacobian corresponding to the uncontrolled switchings remains valid and useful [16]. In particular, we expect the Jacobian simplification for ideal thyristors to continue to be useful in studying damping, resonance and dynamics in high power thyristor controlled reactor circuits for static VAR control and flexible ac transmission [5], [14]–[17], [26] even as the circuits become more complex.

Appendix A

Let $\lambda \in \mathbf{R}^m$ be a parameter and $f\colon \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}$ be the diode current $f(t, \lambda)$ assuming that the diode does not turn off. The switching time $s_{\text{off}}(\lambda) = \min\{s \mid f(s, \lambda) = 0 \text{ and } s > 0\}$.

Lemma A1: If f is a smooth function and the diode switch off at time $s_{\text{off}}(\lambda_1)$ satisfies the transversality condition $\frac{\partial f}{\partial t}(s_{\text{off}}(\lambda_1)) < 0$, then there is an $\epsilon > 0$ such that $s_{\text{off}}(\lambda)$ is a smooth function of λ for $\lambda \in B(\epsilon)$, where $B(\epsilon)$ is the open ball of radius ϵ and center λ_1.

Proof: The implicit function theorem and $\frac{\partial f}{\partial t}(s_{\text{off}}(\lambda_1)) < 0$ imply that there is an $\epsilon_1 > 0$ and a smooth function s: $B(\epsilon_1) \to \mathbf{R}$ such that $s(\lambda_1) = s_{\text{off}}(\lambda_1)$ and $f(s(\lambda), \lambda) = 0$. $\frac{\partial f}{\partial t}(s(\lambda_1)) < 0$ and the continuity of $\frac{\partial f}{\partial t}$ and s imply that there is a $\delta > 0$ and an ϵ_2 with $0 \leq \epsilon_2 \leq \epsilon_1$ such that $\frac{\partial f}{\partial t}(s(\lambda), \lambda) < -\delta < 0$ for $\lambda \in \overline{B(\epsilon_2)}$. By the definition of derivative, for $\lambda \in \overline{B(\epsilon_2)}$, we have $|\frac{1}{h} f(s(\lambda) - h, \lambda) + \frac{\partial f}{\partial t}(s(\lambda), \lambda)| \to 0$ as $h \to 0$. Since $\frac{1}{h} f(s(\lambda) - h, \lambda) + \frac{\partial f}{\partial t}(s(\lambda), \lambda)$ is continuous in λ and $\overline{B(\epsilon_2)}$ is compact, the convergence is uniform for $\lambda \in \overline{B(\epsilon_2)}$. That is, there is an h_0 independent of λ such that $0 < |h| < 2h_0$ implies that $|\frac{1}{h} f(s(\lambda) - h, \lambda) + \frac{\partial f}{\partial t}(s(\lambda), \lambda)| < \frac{\delta}{2}$ for $\lambda \in \overline{B(\epsilon_2)}$. It follows using $\frac{\partial f}{\partial t}(s(\lambda), \lambda) < -\delta < 0$ that $\frac{1}{h} f(s(\lambda) - h, \lambda) > 0$ and hence that $\operatorname{sign}\{h\} f(s(\lambda) - h, \lambda) > 0$ for $\lambda \in \overline{B(\epsilon_2)}$ and h with $0 < |h| < 2h_0$. Now use the continuity of s to choose ϵ_3 with $0 < \epsilon_3 < \epsilon_2$ such that $|s(\lambda) - s(\lambda_1)| < h_0$ for $\lambda \in B(\epsilon_3)$. Then it follows that for $\lambda \in B(\epsilon_3)$ the unique root of f in $[s(\lambda_1) - h_0, s(\lambda_1) + h_0]$ is $s(\lambda)$.

The definition of s_{off} implies that $f(t, \lambda_1) > 0$ for $t \in [0, s_{\text{off}}(\lambda_1) - h_0]$. Write $m = \min\{f(t, \lambda_1) | \, t \in [0, s_{\text{off}}(\lambda_1) - h_0]\}$ and observe that $m > 0$. The uniform continuity of f on $[0, s_{\text{off}}(\lambda_1) - h_0]$ implies that there is an ϵ with $0 < \epsilon < \epsilon_3$ such that for $\lambda \in B(\epsilon)$, $|f(t, \lambda) - f(t, \lambda_1)| < m/2$ and hence that $f(t, \lambda) > 0$ for $t \in [0, s_{\text{off}}(\lambda_1) - h_0] = [0, s(\lambda_1) - h_0]$. This result and the result of the previous paragraph imply that the unique root of f for $t \in [0, s(\lambda_1) + h_0]$ is $s(\lambda)$ if $\lambda \in B(\epsilon)$. That is, for $\lambda \in B(\epsilon)$, $s(\lambda)$ is indeed the first positive root of f and $s_{\text{off}}(\lambda) = s(\lambda)$ is a smooth function of λ. □

APPENDIX B

This appendix constructs nice coordinates x for a circuit with a diode on for which $c = d^T$ and $|x|^2$ is the circuit energy. Standard methods [18], [28], of which we assume some familiarity, yield the following differential equations for the circuit when the diode is on

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{C} \end{pmatrix} \frac{d}{dt} \begin{pmatrix} i_\ell \\ v_c \end{pmatrix} = \begin{pmatrix} -\mathcal{Z} & -\mathcal{H}^T \\ \mathcal{H} & -\mathcal{Y} \end{pmatrix} \begin{pmatrix} i_\ell \\ v_c \end{pmatrix} + \mathcal{B}u. \tag{B1}$$

The state vector consists of inductor link currents i_ℓ and capacitor branch voltages v_c. $\mathcal{L}$ is an inductance matrix, $\mathcal{C}$ is a capacitance matrix, and $\mathcal{Y}$, $\mathcal{H}$, $\mathcal{Z}$ describe a resistive multiport associated with the circuit. $\mathcal{L}$, $\mathcal{C}$, $\mathcal{Y}$, and $\mathcal{Z}$ are symmetric and positive semidefinite [18], [28]. We assume that $\mathcal{L}$ and $\mathcal{C}$ are positive definite to guarantee their invertibility.

Now we consider the rederivation of the circuit differential equations when the diode is off and treated as a voltage source $-v_d$ to make clear the way in which the off diode contributes additional terms to these equations. The first step is to choose a normal tree; that is, the tree links contain the maximum number of inductances, and the tree branches contain the maximum number of capacitors. We claim that the normal tree can be chosen so that the diode is a tree branch and the inductors in the diode cutset κ are links. For if the diode is chosen as a link, then its associated loop must contain one of the inductors in κ as a branch. Then the tree links do not contain the maximum number of inductors because modifying the tree by making the branch inductor a link and the diode a branch increases the number of inductor links by one. Therefore, the diode is a branch. Moreover, the only circumstance in which an inductor in κ is a branch is that it is contained in a cutset of all inductors. Since the cutset of all inductors must be distinct from the cutset κ (κ contains the diode), at least one inductor in the cutset must not be an element of κ, and we can modify the tree to make the inductor not in κ a branch and the inductor in κ a link. Therefore, the normal tree can be chosen so that all the inductors in κ are links.

Applying KCL to the cutset κ yields the current i_d in the diode branch in terms of components of the state

$$i_d = \sum_{\ell_j \in \kappa} i_{\ell_j} = \tilde{c} \begin{pmatrix} i_\ell \\ v_c \end{pmatrix} + \tilde{b}u. \tag{B2}$$

Now use Assumption 2.1 and assume that all the inductor links in κ are oriented in the same direction to ensure that the row vector $\tilde{c}$ contains ones in positions corresponding to inductor links in κ and zeros elsewhere.

The circuit differential equations are obtained by applying KVL to the loops associated with the inductor links and KCL to the cutsets associated with capacitor branches. Then the resistance branch voltages in the equations are expressed in terms of the state by applying KCL to cutsets associated with the resistance branches, and the link conductance currents in the equations are expressed in terms of the state by applying KVL to loops associated with the conductance links.

For inductor links in the cutset κ, the associated loop must include the diode so that the diode voltage appears in that equation. The diode is included in the loop exactly once and with the same orientation for each of the inductor links. For inductor links not in the cutset κ, the associated loop does not include the diode so that the diode voltage does not appear in the corresponding equations. Moreover, the resistance branch voltages in any of these equations do not depend on the diode voltage because the cutset associated with each resistance cannot contain the diode branch. For capacitor branches, the associated cutset cannot contain the diode. Moreover, the conductance link currents do not depend on the diode voltage because the loop associated with each conductance cannot contain the diode branch. (Any loop containing the diode branch must also include an inductor link in κ.)

Formulating the circuit equations as above shows that when the diode is off, the diode voltage v_d appears as an additional term in (B1) according to

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{C} \end{pmatrix} \frac{d}{dt} \begin{pmatrix} i_\ell \\ v_c \end{pmatrix} = \begin{pmatrix} -\mathcal{Z} & -\mathcal{H}^T \\ \mathcal{H} & -\mathcal{Y} \end{pmatrix} \begin{pmatrix} i_\ell \\ v_c \end{pmatrix} + \mathcal{B}u - \tilde{c}^T v_d. \tag{B3}$$

That is, the off diode only contributes a term v_d to each of the equations corresponding to inductive links in the cutset κ. Now change coordinates according to

$$x = M \begin{pmatrix} i_\ell \\ v_c \end{pmatrix} \quad \text{where} \quad M = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{L}^{\frac{1}{2}} & 0 \\ 0 & \mathcal{C}^{\frac{1}{2}} \end{pmatrix}.$$

M is symmetric and positive definite since $\mathcal{L}$ and $\mathcal{C}$ are assumed invertible. Equation (B3) become

$$\dot{x} = Ax + Bu - c^T v_d \tag{B4}$$

where

$$A = (M^T)^{-1}\begin{pmatrix} -\mathcal{Z} & -\mathcal{H}^T \\ \mathcal{H} & -\mathcal{Y} \end{pmatrix} M^{-1},$$
$$B = (M^T)^{-1}\mathcal{B}, \quad c = \tilde{c}M^{-1}. \tag{B5}$$

The positive semidefiniteness of $\mathcal{Y}$ and $\mathcal{Z}$ and the form of (B5) imply that A is negative semidefinite. For dissipative circuits, A is negative definite. Moreover, in the nice coordinates (B2) becomes

$$i_d = cMM^{-1}x + \tilde{b}u = cx + \tilde{b}u. \tag{B6}$$

Comparing (B4) with (2.6) and (B6) with (2.2), it is clear that $c = d^T$ in the nice coordinates. Moreover, in the nice coordinates, the square of the Euclidean norm is

$$|x|^2 = x^T x = (i_\ell^T v_c^T) M^T M \begin{pmatrix} i_\ell \\ v_c \end{pmatrix}$$
$$= \frac{1}{2} i_\ell^T \mathcal{L} i_\ell + \frac{1}{2} v_c^T \mathcal{C} v_c$$

so that $|x|^2$ is the stored energy in the circuit inductors and capacitors [18].

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