Sensitivity of Transient Stability Critical Clearing Time

Shikha Sharma, Sai Pushpak, Venkatesh Chinde, Ian Dobson

Abstract—Once the critical clearing time of a fault leading to transient instability has been computed, it is desirable to quantify its dependence on system parameters. We derive for a general power system model a new and exact formula for the first order sensitivity of the critical clearing time with respect to any system parameter. The formula is evaluated by integrating variational equations forward in time along the base case fault-on trajectory and integrating adjoint variational equations backward in time along the post-fault trajectory. The computation avoids recomputing the critical clearing time for each parameter change and gives insight into how parameters influence power system transient stability. The computation of the sensitivity of the critical clearing time with respect to load impedances and generator inertias is illustrated on a 39-bus system.

Index Terms—Power system transient stability, Numerical integration, Nonlinear dynamical systems

I. Introduction

To maintain transient stability, a power system fault must be cleared quickly enough so that the fault-on transient remains inside the stability boundary. The critical clearing time is the maximum such clearing time, and if the critical clearing time is exceeded, stability is lost by generators losing synchronism. An exact computation of critical clearing time requires numerical integration of fault-on and post-fault trajectories and identification of the controlling unstable equilibrium point that determines the relevant portion of the stability boundary. Critical clearing time is a well-established engineering metric of transient stability and its exact computation by nonlinear analysis and numerical integration [1]–[3] and its approximate computation by energy methods [4]–[13] has been extensively studied.

After critical clearing time has been computed by numerical integration for a base case, it is desirable to evaluate how changing the base case parameters affects the critical clearing time. The influential parameters drive the input data requirements, give insight into what affects transient stability, and guide the engineering to increase the critical clearing time if it is too short.

One way to approach critical clearing time sensitivity is by brute force numerical differencing [14]–[16]. That is, the critical clearing time is recalculated with the parameter changed to evaluate the change in the critical clearing time. For example, Khan [17] analyzes the effect of a variety of parameter changes on the critical clearing time of a single machine infinite bus system by direct simulation. In this paper, we avoid this time-consuming recalculation by analyzing the first-order sensitivity of the critical clearing time to parameters and taking advantage of the efficient power system trajectory

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sensitivity computations pioneered by Hiskens [18], [19]. The power system model assumed for our calculation is general and widely applicable, only requiring that the fault-on and post fault systems be described by smooth, index one, semi-explicit differential algebraic equations.

Several authors approximately reduce multimachine systems to a single machine system to facilitate analysis. Ayasun [20] expresses the critical clearing time of a one machine infinite bus power system model in terms of the load power using the equal area criterion and then linearizes to obtain the first order sensitivity of the critical clearing time with respect to system load. Ayasun points out the importance of the sensitivity of the critical clearing time for probabilistic transient stability assessment, and uses the sensitivity to compute the probability density function of the critical clearing time. Ayasun assumes that the multi-machine systems has first been reduced to a one machine infinite bus power system, whereas our paper directly computes the critical clearing time sensitivity for the general multi-machine case. Trajectory sensitivities along the fault-on trajectory are also used by Xu [21] in a one machine infinite bus equivalent of a larger power system to devise preventive controls to limit angle deviations to stabilize the system.

The functional dependence of the critical clearing time on parameters can also be approximated from numerically obtained samples. Chiodo and Lauria [22] use the extended equal area criterion on a 6-bus 3-generator power system to sample the critical clearing times under variations of the 3 loads. The functional dependence of the logarithm of the critical clearing time on the loads is then obtained by linear regression. A multivariate Gaussian model for load power then leads to a lognormal distribution of critical clearing time to enable a probabilistic evaluation of transient stability.

In previous work, trajectory sensitivities have been used to approximately estimate the critical clearing time. Laufenberg [23] and Nguyen [10] numerically compute trajectory sensitivities in the post-fault system and associate the maximum size of the sensitivity trajectory to the proximity to the stability boundary. In particular, Nguyen et al. compute the sensitivities of machine angles and speeds to the clearing time by computing the trajectory sensitivity forward in time along a fault-on trajectory and further forward in time along the subsequent post-fault trajectory. They note that these sensitivities become large during the post-fault trajectory as the clearing time approaches the critical clearing time, and therefore use the maximum norm of all the sensitivities as an indicator to estimate the critical clearing time. The high sensitivity is caused by the unstable equilibrium point, but the unstable equilibrium point does not need to be explicitly located. While Nguyen's calculation also exploits the trajectory sensitivity techniques of [18] along the fault on and post-fault trajectories, it differs from this paper in evaluating the

critical clearing time, not the sensitivity of the critical clearing time, and using the trajectory sensitivity forward in time of the post-fault differential equations, whereas we apply trajectory sensitivities to the adjoint differential equations of the post-fault system integrated backward in time. Nguyen [10] also uses trajectory sensitivities while evaluating the sensitivity of the energy function evaluated at the controlling unstable equilibrium point. Nguyen [10] then extrapolates two samples of these sensitivities at different clearing times to estimate the critical clearing time.

In this paper we exploit trajectory sensitivities in a novel way to give an exact and general formula for first-order sensitivity of the critical clearing time. The exact formula and its derivation are new, and include integration backward in time of an adjoint variational equation along the post-fault trajectory. After giving a pictorial overview in Section II, Section III describes the power system model. The new sensitivity formulas are derived in Section IV, and their numerical application is described in Section V. A 39-bus example of the calculation is presented in Section VI and Section VII concludes the paper.
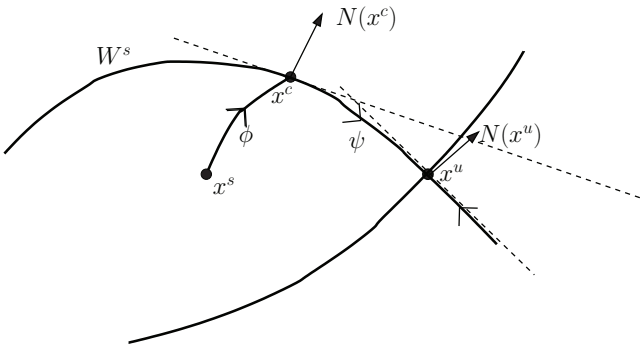
II. PICTORIAL OVERVIEW

Fig. 1. Visualization of the base case in two dimensions. Fault-on trajectory ϕ is from the stable equilibrium x^s to x^c on the stability boundary W^s. Post-fault trajectory ψ in W^s is from x^c to the unstable equilibrium x^u. N is the normal vector to W^s.

We first give an overview of the computation in the two-dimensional case that is easiest to depict. (Readers familiar with the geometry of the nonlinear dynamics of transient stability may skip this overview.) The equilibria and trajectories to be computed in the base case are shown in Fig. 1. The stable equilibrium of the pre-fault system is x^s, and the trajectory ϕ is integrated from x^s in the fault-on system until it reaches the stability boundary W^s at x^c at the critical clearing time t^c. The stability boundary W^s is in the post-fault system and is the stable manifold of the unstable equilibrium point x^u. The critical post-fault trajectory ψ starts from x^c and ends at x^u.

In this two dimensional case, the relevant part of the stable manifold W^s coincides with the one dimensional post-fault trajectory ψ. The stable manifold W^s can be approximated near x^c by the dashed tangent line shown in Fig. 1. The normal vector $N(x^c)$ to W^s at x^c determines the inclination of the tangent line. The normal vector $N(x^u)$ of W^s at x^u is a

left eigenvector of the unstable eigenvalue of the linearization at x^u. It will be shown that the normal vector $N(x^c)$ can be obtained by starting with the normal vector $N(x^u)$ at the unstable equilibrium x^u and integrating an adjoint differential equation backward in time along the trajectory ψ.

If a general parameter is changed, all the equilibria and trajectories and x^c change as shown in Fig. 2. The linearized changes in the equilibria x^s and x^u are easily obtained. The linearized change in ϕ can be obtained by integrating a variational fault-on differential equation forward in time from 0 to t^c. The linearized change in the stable manifold W^s can be obtained by integrating a variational differential equation backward in time along ψ. The change in the stable manifold W^s changes its intersection x^c with the fault-on trajectory, causing a change in the final part of the movement along the fault-on trajectory to the new intersection. The linearized change in clearing time caused by the movement along the fault-on trajectory to the new intersection is given by the tangent velocity of the fault-on trajectory at x^c (not shown in Fig. 2), which is given by the fault-on differential equations evaluated at x^c. It is plausible, and proved in the following sections, that suitably combining all these linearized changes yields the sensitivity of the critical clearing time.
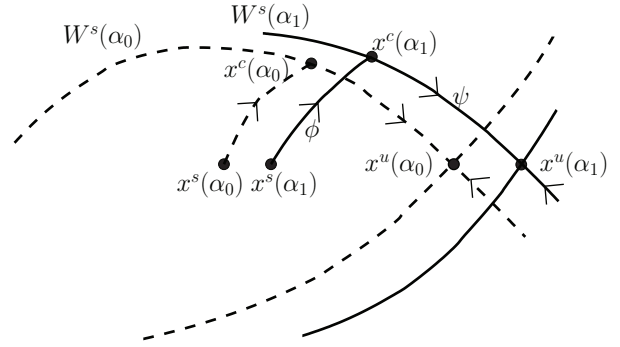

Fig. 2. Change in the equilibrium points, trajectories, and stability boundary for a parameter change from the base case value α_0 to α_1.

We emphasize that Figs. 1 and 2 require additional dimensions to be imagined for the intended higher-dimensional applications. In particular, the stable manifold W^s is a hypersurface. For example, Fig. 3 shows W^s, its normal vector N, and the post-fault trajectory ψ in three dimensions.

III. POWER SYSTEM MODEL

This section specifies fault-on and post-fault power system dynamic models and notation.

A. Fault-on differential equations

The fault-on power system differential equations are

$$\dot{x} = f(x, \alpha) \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state, f is smooth, and α is any parameter.

The general solution to (1) with initial condition x_0 is
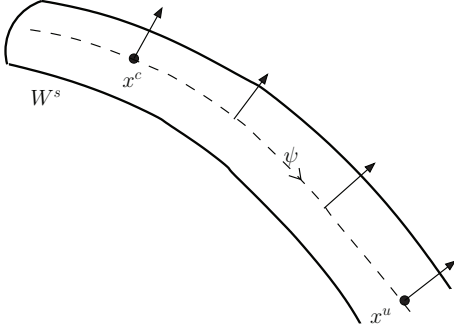
$$\phi(x_0, t, \alpha) \qquad (2)$$

Fig. 3. Post-fault trajectory ψ on the stability boundary W^s and the normal vectors along ψ in three dimensions.

$x^s(\alpha)$ is the pre-fault stable equilibrium and operating point. The base case parameter is α_0 and the pre-fault base case stable equilibrium is $x_0^s = x^s(\alpha_0)$. The nominal fault-on trajectory is $\phi(x_0^s, t, \alpha_0)$ for $t \geq 0$.

B. Post-fault differential equations

The post-fault power system differential equations are

$$\dot{x} = F(x, \alpha) \qquad (3)$$

where F is smooth. The general solution to (3) with initial condition x_0 is

$$\psi(x_0, t, \alpha) \qquad (4)$$

The controlling unstable equilibrium is $x^u(\alpha)$ and the stable manifold of $x^u(\alpha)$ is $W^s(x^u(\alpha))$. $W^s(x^u(\alpha))$ is part of the basin boundary of the post-fault stable operating point $x_{\text{post}}^s(\alpha)$ [1]. In particular, if the fault-on trajectory has not reached $W^s(x^u(\alpha))$ when the fault clears, then the system will restabilize at $x_{\text{post}}^s(\alpha)$. If the fault-on trajectory crosses $W^s(x^u(\alpha))$ before the fault clears, then the system is transiently unstable.

In the theory derivation in this paper, the power system model dynamics are expressed as differential equations to simplify their expression, similarly to [19]. In practice, differential equations (1) and (3) are routinely obtained from index one semi-explicit differential-algebraic equations. The computations can be adapted to apply directly to the differential-algebraic equations [18], as indicated in the appendix.

C. Critical clearing time

The fault starts at time zero at $x^s(\alpha)$. The critical clearing time $t^c(\alpha)$ is the first time that the fault-on trajectory intersects $W^s(x^u(\alpha))$. Write $x^c(\alpha)$ for the first intersection of the fault-on trajectory with $W^s(x^u(\alpha))$:

$$x^c(\alpha) = \phi(x^s(\alpha), t^c(\alpha), \alpha) \qquad (5)$$

Suppose that $W^s(x^u(\alpha))$ has equation

$$S(x, \alpha) = 0 \qquad (6)$$

near $x^c(\alpha)$. A suitable S is defined in section IV-B. Then

$$0 = S(x^c(\alpha), \alpha) = S(\phi(x^s(\alpha), t^c(\alpha), \alpha), \alpha) \qquad (7)$$

In particular,

$$0 = S(x_0^c, \alpha_0) \qquad (8)$$

where $x_0^c = x^c(\alpha_0)$.

D. System critical trajectory

The base case critical trajectory is

$$\begin{cases} \phi(x_0^s, t, \alpha_0), & 0 \leq t < t_0^c & \text{fault-on} \\ \psi(x_0^c, t, \alpha_0), & t_0^c \leq t < \infty & \text{post-fault} \end{cases} \qquad (9)$$

where $t_0^c = t^c(\alpha_0)$. The base case critical trajectory starts at time zero at $x_0^s = \phi(x_0^s, 0, \alpha_0)$, passes through x_0^c at time t_0^c and then tends to $x^u(\alpha_0)$ as time tends to infinity. The critical clearing time t_0^c is chosen to make the base case critical trajectory marginally stable and tend to $x^u(\alpha_0)$.

More generally, accounting for the variation of the critical trajectory with respect to the parameter α, the critical trajectory is

$$\begin{cases} \phi(x^s(\alpha), t, \alpha), & 0 \leq t < t^c(\alpha) & \text{fault-on} \\ \psi(x^c(\alpha), t, \alpha), & t^c(\alpha) \leq t < \infty & \text{post-fault} \end{cases} \qquad (10)$$

The critical trajectory starts at time zero at $x^s(\alpha)$, passes through $x^c(\alpha)$ at time $t^c(\alpha)$ and tends to $x^u(\alpha)$ as time tends to infinity.

When α changes, both $x^s(\alpha)$ and $x^u(\alpha)$ change and this affects the fault-on and post-fault trajectories respectively. In addition, the fault-on and post-fault flows ϕ and ψ directly depend on α. These changes in the fault-on and post-fault trajectories cause the clearing time to change. The sensitivity formula derived below quantifies these dependencies.

IV. SENSITIVITY FORMULA DERIVATION

This section derives the sensitivity formula using variational and dynamical systems methods. A subscripted variable indicates (partial) differentiation of that variable with respect to the subscripted variable and | means "evaluated at".

Differentiating (7) with respect to α yields

$$0 = S_x \left(\phi_x x_\alpha^s + \phi_t t_\alpha^c + \phi_\alpha \right) + S_\alpha \qquad (11)$$

Rearranging and using $\phi_t = f(x^c, \alpha)$ gives

$$t_\alpha^c = - \left(S_x f(x^c, \alpha) \right)^{-1} \left(S_x(\phi_x x_\alpha^s + \phi_\alpha) + S_\alpha \right) \qquad (12)$$

and evaluating at α_0 gives the desired sensitivity formula:

$$\boxed{\begin{aligned} t_\alpha^c|_{\alpha_0} = &- \left(S_x|_{(x_0^c, \alpha_0)} f(x_0^c, \alpha_0) \right)^{-1} \times \\ &\left[S_x|_{(x_0^c, \alpha_0)} (\phi_x|_{(x_0^s, t_0^c, \alpha_0)} x_\alpha^s|_{\alpha_0} + \phi_\alpha|_{(x_0^s, t_0^c, \alpha_0)}) + S_\alpha|_{(x_0^c, \alpha_0)} \right] \end{aligned}}$$

$$(13)$$

The quantities that need to be computed to evaluate the sensitivity formula (13) are:

1) $S_x|_{(x_0^c, \alpha_0)} = N(x_0^c, \alpha_0)$ is a normal vector to the stable manifold $W^s(x^u)$ at x_0^c.
2) $S_\alpha|_{(x_0^c, \alpha_0)}$ is the sensitivity of the stable manifold with respect to α at x_0^c.
3) $\phi_x|_{(x_0^s, t_0^c, \alpha_0)}$ is the sensitivity of the fault-on trajectory with respect to the initial condition $x^s(\alpha)$ at x_0^c.

4) $\phi_\alpha|_{(x_0^s, t_0^c, \alpha_0)}$ is the sensitivity of the fault-on trajectory with respect to α at x_0^c.

5) $x_\alpha^s|_{\alpha_0}$ is the sensitivity of the stable equilibrium with respect to α and is obtained by solving

$$f_x|_{(x_0^s, \alpha_0)}\, x_\alpha^s|_{\alpha_0} = -f_\alpha|_{(x_0^s, \alpha_0)} \tag{14}$$

6) $f(x_0^c, \alpha_0)$ is evaluated directly from the fault-on differential equation (1).

The following subsections derive the first four quantities in the list.

A. Sensitivity of fault-on trajectory

It follows from differential equation (1) that

$$\phi_t|_{(x^s(\alpha), t, \alpha)} = f(\phi(x^s(\alpha), t, \alpha), \alpha) \tag{15}$$

Differentiating (15) with respect to $x^s(\alpha)$ gives

$$\phi_{xt}|_{(x^s(\alpha), t, \alpha)} = f_x|_{(\phi(x^s(\alpha), t, \alpha), \alpha)} \phi_x|_{(x^s(\alpha), t, \alpha)} \tag{16}$$

Evaluation along the fault-on trajectory gives

$$\phi_{xt}|_{(x_0^s, t, \alpha_0)} = f_x|_{(\phi(x_0^s, t, \alpha_0), \alpha_0)} \phi_x|_{(x_0^s, t, \alpha_0)} \tag{17}$$

Integrating (17) along the fault-on trajectory from time zero to t_0^c with initial condition the identity matrix yields $\phi_x|_{(x_0^s, t_0^c, \alpha_0)}$.

Differentiating (15) with respect to α gives

$$\begin{aligned}\phi_{\alpha t}|_{(x^s(\alpha), t, \alpha)} = &f_x|_{(\phi(x^s(\alpha), t, \alpha), \alpha)}(\phi_x|_{(x^s(\alpha), t, \alpha)} x_\alpha^s|_\alpha \\ &+ \phi_\alpha|_{(x^s(\alpha), t, \alpha)}) + f_\alpha|_{(\phi(x^s(\alpha), t, \alpha), \alpha)}\end{aligned} \tag{18}$$

Evaluation along the fault-on trajectory gives

$$\begin{aligned}\phi_{\alpha t}|_{(x_0^s, t, \alpha_0)} = &f_x|_{(\phi(x_0^s, t, \alpha_0), \alpha_0)}(\phi_x|_{(x_0^s, t, \alpha_0)} x_\alpha^s|_{\alpha_0} \\ &+ \phi_\alpha|_{(x_0^s, t, \alpha_0)}) + f_\alpha|_{(\phi(x_0^s, t, \alpha_0), \alpha_0)}\end{aligned} \tag{19}$$

Now $\phi_\alpha|_{(x_0^s, t_0^c, \alpha_0)}$ is calculated by integrating (19) along the fault-on trajectory from time zero to t_0^c with initial condition

$$\phi_\alpha(x_0^s, 0, \alpha_0) = x_\alpha^s|_{\alpha_0} \tag{20}$$

This fault-on variational trajectory calculation is also done in Laufenberg [23] for a 17-bus system, and for an equivalent single machine system in [21].

B. Defining the function S that describes W^s

This subsection uses nonlinear dynamical systems methods to define a function $S(x, \alpha)$ so that the stability boundary and stable manifold $W^s(x^u(\alpha))$ has equation

$$S(x, \alpha) = 0 \tag{21}$$

near the critical trajectory, including near $x^c(\alpha)$. The tools used are standard constructions in nonlinear dynamical systems, but the derivation is new. Suitable background material for these methods is in [24]–[26].

We make the generic assumption that the linearized dynamics at the controlling unstable equilibrium point x_0^u has all eigenvalues with negative real parts except for one eigenvalue that is real and positive. It follows that $x^u(\alpha)$ is a smooth function of α sufficiently near α_0, and that a suitably normalized left eigenvector $w(\alpha)$ corresponding to the unstable eigenvalue is a smooth function of α. We write $w_0 = w(\alpha_0)$.
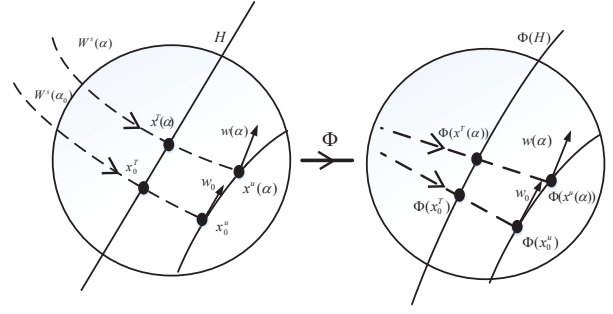

Fig. 4. Detail near the controlling unstable equilibrium point x_0^u and its mapping under Φ to locally flatten the stable manifolds for the construction of the function S. Two stable manifolds are shown by dotted lines, one for the base case parameter value α_0 and the other for the parameter α close to α_0. The 2 dimensional case is shown.

Now we establish new coordinates near $x^u(\alpha)$ with a transformation Φ^α in which $W^s(x^u(\alpha))$ becomes locally a hyperplane passing through $x^u(\alpha)$. Let $B \subset \mathbb{R}^n$ be a small enough ball containing x_0^u, and suppose that α is sufficiently close to α_0. Write $W_{loc}^s(x^u(\alpha)) = W^s(x^u(\alpha)) \cap B$. Let $\Phi^\alpha : B \to \mathbb{R}^n$ be a diffeomorphism for which $\Phi^\alpha(W_{loc}^s(x^u(\alpha)))$ is a hyperplane $E^s(\alpha)$ through $x^u(\alpha)$ for each α, and $\Phi_x^\alpha|_{x^u(\alpha)}$ is the identity matrix I for each α. This follows from a parameterized version of the stable manifold theorem [26][1]. Then $w(\alpha) = w(\alpha)I = w(\alpha)\Phi_x^\alpha|_{x^u(\alpha)}$ is normal to both $W_{loc}^s(x^u(\alpha))$ and $E^s(\alpha)$. In particular, the equation in the variable x of $W_{loc}^s(x^u(\alpha))$ is

$$w(\alpha)[\Phi^\alpha(x) - \Phi^\alpha(x^u(\alpha))] = 0. \tag{22}$$

Let T be a time such that $\psi(x_0^c, T, \alpha_0)$ is in B. (That is, integrate along the critical post-fault trajectory until a time T when the state is near enough x_0^u. We can increase T later if needed.) We write $x^T(\alpha) = \psi(x^c(\alpha), T, \alpha)$ and $x_0^T = x^T(\alpha_0) = \psi(x_0^c, T, \alpha_0)$. Let H be a hyperplane through x_0^T transverse to the post-fault trajectory. The local stable manifold $W_{loc}^s(x^u)$ intersects H in a manifold $W_{loc}^s(x^u) \cap H$ of dimension $n - 2$ near x_0^T.

In some neighborhood of the post-fault critical trajectory, we can define $\tau(x, \alpha)$ as the time for the trajectory starting at x to first reach $H \cap B$. τ generally satisfies

$$\psi(x, \tau(x, \alpha), \alpha) \in H \cap B \tag{23}$$

and, if x is on the base case post-fault trajectory, $\psi(x, \tau(x, \alpha_0), \alpha_0) = x_0^T$.

Now we can define

$$S(x, \alpha) = w(\alpha)\big[\Phi^\alpha(\psi(x, \tau(x, \alpha), \alpha)) - \Phi^\alpha(x^T(\alpha))\big] \tag{24}$$

It follows from (22) and (23) that $W^s(x^u(\alpha))$ satisfies $S(x, \alpha) = 0$ near the post-fault critical trajectory. In essence, S measures the distance of x from the stable manifold $W^s(x^u(\alpha))$ by following the trajectory through x until it hits the hyperplane H near x_0^u and then projecting perpendicular to the local stable manifold $W_{loc}^s(x^u(\alpha))$.

[1] [26, theorem 6.2] applies to a neighborhood of maps, but this adapts to the result for parameterized flows needed here.

Now we discuss the sign of S. There is an ambiguity in the sign of the left eigenvector $w(\alpha)$ in the formula (24) and hence an ambiguity in the sign of S. Since $S(x, \alpha) = 0$ describes part of the stable manifold of $x^u(\alpha)$, it separates transiently stable and unstable trajectories. For definiteness we could now choose the sign of $w(\alpha)$ so that $S(x, \alpha_0) > 0$ for unstable trajectories and $S(x, \alpha_0) < 0$ for stable trajectories. However, any consistent sign for $w(\alpha)$ and S can be used, since the sensitivity formula (13) does not depend on the sign of S.

The way that $S(x, \alpha)$ is defined by first moving along the trajectory through x until it meets H also ensures that S is invariant along trajectories. This can be shown explicitly as follows:

$$
\begin{aligned}
S(&\psi(x, t, \alpha), \alpha) \\
&= w(\alpha)\big[\Phi^\alpha\big(\psi(\psi(x, t, \alpha), \tau(\psi(x, t, \alpha), \alpha), \alpha)\big) \\
&\quad - \Phi^\alpha\big(x^T(\alpha)\big)\big] \quad (25) \\
&= w(\alpha)\big[\Phi^\alpha\big(\psi(\psi(x, t, \alpha), \tau(x, \alpha) - t, \alpha), \alpha)\big) \\
&\quad - \Phi^\alpha\big(x^T(\alpha)\big)\big] \quad (26) \\
&= w(\alpha)\big[\Phi^\alpha\big(\psi(x, \tau(x, \alpha), \alpha)\big) - \Phi^\alpha\big(x^T(\alpha)\big)\big] \quad (27) \\
&= S(x, \alpha) \quad (28)
\end{aligned}
$$

Equation (25) follows from the definition of S in (24), (26) follows since the time for an initial point to reach H along its trajectory is reduced by t if the initial point is moved for time t along its trajectory, (27) follows from the basic property of differential equations that moving along a trajectory for time t, and then for time $\tau - t$ has the same result as moving along a trajectory for time τ, and (28) recalls the definition of S.

C. Computing the stable manifold normal vector S_x with adjoint variational equations

This subsection computes the stable manifold normal vector $S_x|_{(x_0^c, \alpha_0)}$ by integrating equations adjoint to the post-fault variational equations backward in time.

It follows from the post-fault differential equation (3) that

$$
\psi_t|_{(x_0(\alpha), t, \alpha)} = F(\psi(x_0(\alpha), t, \alpha), \alpha) \quad (29)
$$

Differentiating (29) with respect to $x_0(\alpha)$ gives

$$
\psi_{xt}|_{(x_0(\alpha), t, \alpha)} = F_x|_{(\psi(x_0(\alpha), t, \alpha), \alpha)}\psi_x|_{(x_0(\alpha), t, \alpha)} \quad (30)
$$

Differentiating (29) with respect to α gives

$$
\begin{aligned}
\psi_{\alpha t}|_{(x_0(\alpha), t, \alpha)} = &F_x|_{(\psi(x_0(\alpha), t, \alpha), \alpha)}\big(\psi_x|_{(x_0(\alpha), t, \alpha)}x_{0\alpha}|_\alpha \\
&+ \psi_\alpha|_{(x_0(\alpha), t, \alpha)}\big) + F_\alpha|_{(\psi(x_0(\alpha), t, \alpha), \alpha)} \quad (31)
\end{aligned}
$$

It is convenient to temporarily omit the dependence on α from the notation to reduce clutter. The invariance of S along trajectories (28) becomes in the case of the fault-on trajectory

$$
S(\psi(x, t)) = S(x) \quad (32)
$$

Differentiating with respect to x,

$$
S_x|_{\psi(x,t)}\psi_x|_{(x,t)} = S_x|_x \quad (33)
$$

Differentiating with respect to t gives

$$
\left(\frac{d}{dt}S_x|_{\psi(x,t)}\right)\psi_x|_{(x,t)} + S_x|_{\psi(x,t)}\psi_{xt}|_{(x,t)} = 0 \quad (34)
$$

and (30) gives

$$
\left(\frac{d}{dt}S_x|_{\psi(x,t)}\right)\psi_x|_{(x,t)} = -S_x|_{\psi(x,t)}F_x|_{\psi(x,t)}\psi_x|_{(x,t)} \quad (35)
$$

Since $\psi_x|_{(x,t)}$ is invertible,

$$
\frac{d}{dt}S_x|_{\psi(x,t)} = -S_x|_{\psi(x,t)}F_x|_{\psi(x,t)} \quad (36)
$$

and evaluating on the base case critical trajectory and restoring the dependence on α gives

$$
\begin{aligned}
\frac{d}{dt}S_x|_{(\psi(x_0^T, t-T, \alpha_0), \alpha_0)} = \\
- S_x|_{(\psi(x_0^T, t-T, \alpha_0), \alpha_0)}F_x|_{(\psi(x_0^T, t-T, \alpha_0), \alpha_0)}. \quad (37)
\end{aligned}
$$

The initial condition is $S_x|_{(x_0^T, \alpha_0)} = w$ at $t = T$ and integrating (37) backward in time from $t = T$ to $t = t_0^c$ yields $S_x|_{(\psi(x_0^T, t_0^c - T, \alpha_0), \alpha_0)} = S_x|_{(x_0^c, \alpha_0)}$ Note that (37) is the differential equation *adjoint* to (30) [27].

D. Computing $S_\alpha|_{(x_0^c, \alpha_0)}$

The invariance of S (28) on a trajectory through $x_0(\alpha)$ gives

$$
S(\psi(x_0(\alpha), t, \alpha), \alpha) = S(x_0(\alpha), \alpha) \quad (38)
$$

Differentiating (38) with respect to α,

$$
\begin{aligned}
S_x|_{(\psi(x_0(\alpha), t, \alpha), \alpha)}&\big(\psi_x|_{(x_0(\alpha), t, \alpha)}x_{0\alpha}|_\alpha + \psi_\alpha|_{(x_0(\alpha), t, \alpha)}\big) + \\
&S_\alpha|_{(\psi(x_0(\alpha), t, \alpha), \alpha)} \\
&= S_x|_{(x_0(\alpha), \alpha)}x_{0\alpha}|_\alpha + S_\alpha|_{(x_0(\alpha), \alpha)}x_{0\alpha}|_\alpha \quad (39)
\end{aligned}
$$

Differentiating (39) with respect to t,

$$
\begin{aligned}
&\left(\frac{d}{dt}S_x|_{(\psi(x_0(\alpha), t, \alpha), \alpha)}\right)\big(\psi_x|_{(x_0(\alpha), t, \alpha)}x_{0\alpha}|_\alpha + \psi_\alpha|_{(x_0(\alpha), t, \alpha)}\big) \\
&+ S_x|_{(\psi(x_0(\alpha), t, \alpha), \alpha)}\big(\psi_{xt}|_{(x_0(\alpha), t, \alpha)}x_{0\alpha}|_\alpha + \psi_{\alpha t}|_{(x_0(\alpha), t, \alpha)}\big) \\
&+ \frac{d}{dt}S_\alpha|_{(\psi(x_0(\alpha), t, \alpha), \alpha)} = 0 \quad (40)
\end{aligned}
$$

or, more briefly,

$$
\frac{d}{dt}S_\alpha + \left(\frac{d}{dt}S_x\right)(\psi_x x_{0\alpha} + \psi_\alpha) + S_x(\psi_{xt} x_{0\alpha} + \psi_{\alpha t}) = 0
$$

Using (37), (30) and (31),

$$
\begin{aligned}
\frac{d}{dt}S_\alpha &- S_x F_x(\psi_x x_{0\alpha} + \psi_\alpha) + \\
&S_x(F_x\psi_x x_{0\alpha} + F_x(\psi_x x_{0\alpha} + \psi_\alpha) + F_\alpha) = 0 \quad (41)
\end{aligned}
$$

so that

$$
\frac{d}{dt}S_\alpha + S_x F_x \psi_x x_{0\alpha} + S_x F_\alpha = 0 \quad (42)
$$

or,

$$
\begin{aligned}
\frac{d}{dt}S_\alpha|_{(\psi(x_0(\alpha), t, \alpha), \alpha)} = &-S_x|_{(\psi(x_0(\alpha), t, \alpha), \alpha)}F_\alpha|_{(\psi(x_0(\alpha), t, \alpha), \alpha)} \\
&- S_x|_{(\psi(x_0(\alpha), t, \alpha), \alpha)}F_x|_{(\psi(x_0(\alpha), t, \alpha), \alpha)}\psi_x|_{(x_0(\alpha), t, \alpha)}x_{0\alpha}|_\alpha \quad (43)
\end{aligned}
$$

Evaluate (43) on the base case post-fault critical trajectory to get

$$
\begin{aligned}
\frac{d}{dt}S_\alpha|_{(\psi(x_0^T, t-T, \alpha_0), \alpha_0)} = \\
- S_x|_{(\psi(x_0^T, t-T, \alpha_0), \alpha_0)}F_x|_{(\psi(x_0^T, t-T, \alpha_0), \alpha_0)}\psi_x|_{(x_0^T, t-T, \alpha_0)}x_\alpha^T|_{\alpha_0} \\
- S_x|_{(\psi(x_0^T, t-T, \alpha_0), \alpha_0)}F_\alpha|_{(\psi(x_0^T, t-T, \alpha_0), \alpha_0)} \quad (44)
\end{aligned}
$$

evaluation of the Jacobian $F_x|_{(\psi(x_0^T,t-T,\alpha_0),\alpha_0)}$ along the post-fault trajectory.

10) Compute $S_\alpha|_{(x_0^c,\alpha_0)}$ by integrating (11) backward in time from $t = T$ to $t = t_0^c$ from initial condition $S_\alpha|_{(x_0^T,\alpha_0)}$. This requires the evaluation of $F_x|_{(\psi(x_0^T,t-T,\alpha_0),\alpha_0)}$, $F_\alpha|_{(\psi(x_0^T,t-T,\alpha_0),\alpha_0)}$, and $\psi_x|_{(x_0^T,t-T,\alpha_0)}$ along the post-fault trajectory.

11) Compute the sensitivity $t_\alpha^c|_{\alpha_0}$ using formula (13).

VI. POWER SYSTEM EXAMPLE

This section considers the 39-bus 10-generator power system shown in Fig. 5 to illustrate the computation of the sensitivity of critical clearing time with respect to load impedances and generator inertias. The bus and system parameter values are taken from [30].
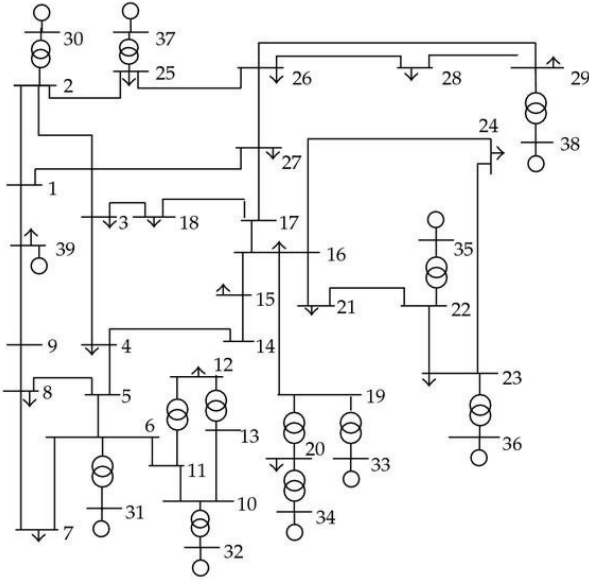

Fig. 5. 39 bus example system. A three phase ground fault occurs on the line between bus 17 and bus 18.

The generator at bus i has fifth order dynamics representing the swing, two axis flux, and field voltage dynamics:

$$\dot{\delta}_i = \omega_i - \omega_s, \tag{48}$$

$$\dot{\omega}_i = \frac{\omega_s}{2H_i^\alpha}[P_{m_i} - d_i(\omega_i - \omega_s)$$
$$- V_i I_{qi}\cos(\theta_i - \delta_i) + V_i I_{di}\sin(\delta_i - \theta_i)] \tag{49}$$

$$T_{qi}\dot{E}_{qi}' = -E_{qi}' + (x_{di} - x_{qi}')I_{di} + E_{fdi} \tag{50}$$

$$T_{di}\dot{E}_{di}' = -E_{di}' + (x_{qi} - x_{qi}')I_{qi} \tag{51}$$

$$T_{Ai}\dot{E}_{fdi}' = -E_{fdi}' + K_{Ai}(V_{\text{ref},i} - V_i) \tag{52}$$

for $i = 30, 31, ..., 39$. ω_s is the synchronous speed, P_{m_i} is the constant mechanical power input and $P_{e_i} = \text{Re}(V_i e^{\theta_i} I_i e^{-\phi_i})$. The current injection at each generator bus is computed from $I_i\angle\phi_i = I_{di} + jI_{qi}$, $i = 30, 31, ..., 39$. The bus phasor voltage vector V is computed from the network equations $V = Y_{bus}^{-1} I$.

α_i^Z is the load impedance parameter. α_i^Z enters the equations via the Y_{bus} matrix entry; e.g., for load at bus 15:

$$Y_{15,15} = \frac{1}{Z_{15,14}} + \frac{1}{Z_{15,16}} + \frac{1}{Z_{\text{load15}} + \alpha_{15}^Z} \tag{53}$$

α^H is the generator inertia parameter, and it enters (49) according to

$$H_i^\alpha = H_i + \alpha_i^H, \quad i = 30, 31, ..., 39. \tag{54}$$

Although inertia is a constant for any given generator, here it can be a parameter since the generator is an equivalent lumped model of group of generators. Indeed, decrease in lumped inertia is a growing concern as inverter-based generation sources displace spinning generators.

The preliminary base case computations are now summarized. A three phase ground fault is introduced at time zero between bus 17 and bus 18 at 200 km from bus 18 and the fault-on critical trajectory is computed. The base case critical clearing time t_0^c of 0.34 s, x^c, and the corresponding post-fault trajectory starting at 0.34 s are computed using the shooting method. The relative rotor angles of the critical trajectory are shown in Fig. 6. This critical trajectory needs to be computed to determine the base case critical clearing time. Variations around this critical trajectory (the fault-on portion forward in time and the post-fault portion backward in time) are central to computing the sensitivity of the critical clearing time.

Fig. 6. Relative rotor angles of the critical trajectory. This critical trajectory is calculated before the sensitivity calculation that linearizes deviations about this trajectory is done.

To compute the sensitivity of the critical clearing time with respect to load impedances α^Z and generator inertias α^H as parameters, we use MATLAB to evaluate all the steps in Section V and hence evaluate formula (13). All the numerical integrations use the MATLAB ode15s solver. The time-varying matrix variational differential equations (16) have the overall form $\dot{X} = M(t)X$. They are numerically integrated by converting $\dot{X} = M(t)X$ to a vector differential equation. The columns of the $m \times m$ state matrix X are stacked into a vector x of length m^2, and the vector differential equation is $\dot{x} = A(t)x$, where the $A(t)$ is the block diagonal $m^2 \times m^2$ matrix diag$\{M(t), M(t), ..., M(t)\}$.

We first focus on the sensitivities to two parameters, the impedance of load 15 and the inertia of generator 33. The critical clearing time first-order sensitivities computed with

formula (13) are

$$\left.\frac{\partial t^c}{\partial \alpha^Z}\right|_{\alpha^Z=0} = t^c_{\alpha^Z_{15}}|_0 = -5.3816 \quad \text{s/pu} \qquad (55)$$

$$\left.\frac{\partial t^c}{\partial \alpha^H}\right|_{\alpha^H=0} = t^c_{\alpha^H_{33}}|_0 = 0.1256 \quad \text{s/pu} \qquad (56)$$

These critical clearing time sensitivities can be used in a linearized model relating the clearing time to the parameter change $\Delta\alpha = \alpha - \alpha_0$ relative to the base case parameter α_0:

$$t^c(\alpha) = t^c_0 + t^c_\alpha|_0 \Delta\alpha \qquad (57)$$

where t^c_0 is the base case critical clearing time.

To confirm the sensitivity calculation with the linearization (57), we also computed the actual critical clearing time t^c as a function of the parameter by brute-force re-computing t^c as the parameter varies. The actual critical clearing times and the linearized critical clearing times computed from the formula (13) and the linearization (57) are shown for each parameter in Figs. 7 and 8. The tangency of the dashed and solid lines confirms the correctness of the computation of the sensitivity of the critical clearing time with formula (13). Figs. 7 and 8 also show the mild nonlinearity of the critical clearing time with respect to load 15 impedance and generator 33 inertia.

To show one way in which the sensitivities can be applied, compare the dependence on twenty parameters of the critical clearing time based on sensitivities with the actual critical clearing times in Figs. 9,10 and in Figs. 11,12. It is clear that the sensitivities can be used to select the parameters that affect the critical clearing time the most, and approximately quantify this dependence.

The computation of sensitivity of the critical clearing time is performed on a 2.4 GHz Intel Core i7 processor in the MATLAB R2017b environment. The major effort of the calculations is integrating variational equations (both forward and backward). For our 39-bus simulation example, the overall computation time including the base case computations is 28 s, and the new sensitivity computations by themselves are 9 s per parameter. (The base case computations of 19 s include finding the base case critical trajectory and its base case critical clearing time.) The calculation related to the inverse of S_x is also of main interest when addressing the computational effort of the sensitivity of critical clearing time.

VII. CONCLUSION

Given an exact calculation of the transient stability critical clearing time with its associated critical trajectories, we derive a new formula for the first order sensitivity of the critical clearing time with respect to any power system parameter and show how to numerically evaluate the formula using trajectory sensitivities. The formula and its derivation are novel in power systems analysis. The new formula is exact but its evaluation requires numerical methods. The formula is general and is widely applicable to power system differential-algebraic models for the fault-on and post-fault systems.

The computations include a conventional variational equation evaluated along the fault-on critical trajectory and a novel adjoint variational equation evaluated backward in time along the post-fault critical trajectory. Both the normal vector to the

Fig. 7. Critical clearing time t^c as a function of load impedance parameter α^Z estimated using the sensitivity of the critical clearing time (solid line) and the actual critical clearing time (dashed curve).

Fig. 8. Critical clearing time t^c as a function of generator 33 inertia parameter α^H estimated using the sensitivity of the critical clearing time (solid line) and the actual critical clearing time (dashed curve).
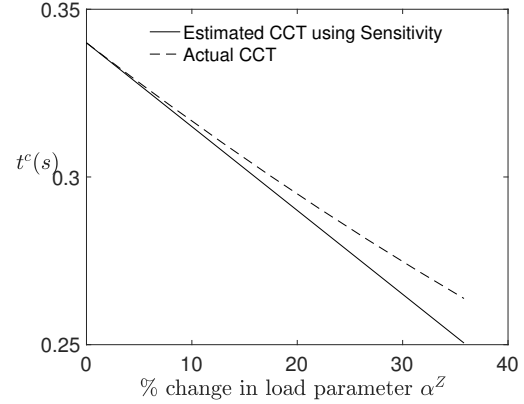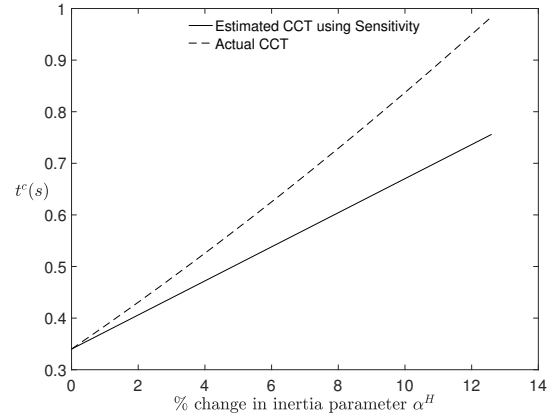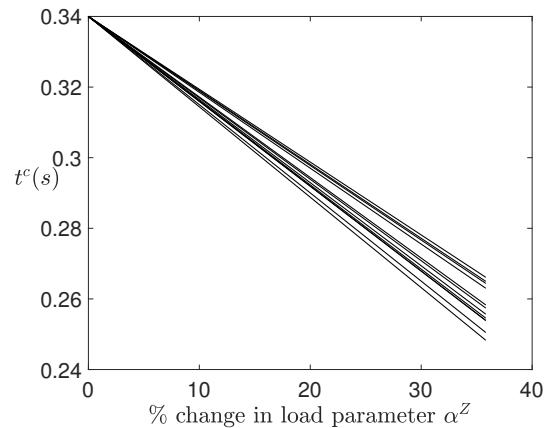

Fig. 9. Critical clearing time t^c as a function of eleven load impedance parameters α^Z estimated using the sensitivity of the critical clearing time.
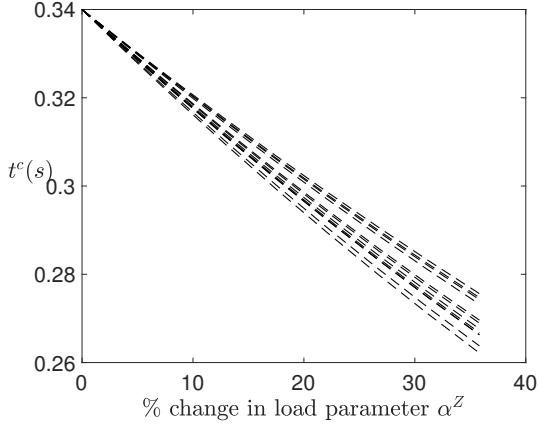
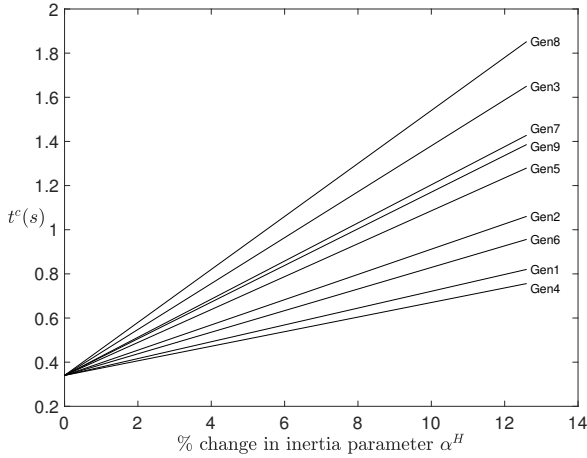Fig. 10. Actual critical clearing time t^c as a function of eleven load impedance parameters α^Z.

Fig. 11. Critical clearing time t^c as a function of nine generator inertia parameters α^H estimated using the sensitivity of the critical clearing time.
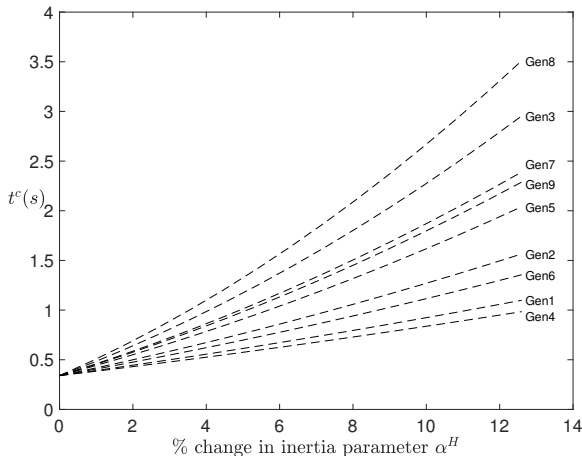

Fig. 12. Actual critical clearing time t^c as a function of nine generator inertia parameters α^H.

stability boundary hypersurface and the first order variation of the stability boundary with respect to the parameter are propagated backward in time with the adjoint variational equation, and this is a new method in power systems analysis. More generally in computational nonlinear dynamics, it is challenging to compute with higher dimensional stable manifolds because of the complexities of tracking hypersurfaces in higher dimensions [31]. Our computation avoids such difficulties by computing the adjoint variational equation along the one-dimensional post-fault trajectory that lies in the stable manifold. More generally, our computation leverages Hiskens' efficient trajectory sensitivities calculations [18] and applies nonlinear dynamics to give a new calculation of the first order sensitivity of a classical metric of transient stability.

Computing the first-order sensitivity of the critical clearing time avoids the tedious brute-force recomputation of the clearing time with parameters varying from the base case while quantifying how much various parameters affect the critical clearing time. Insight into which parameters strongly influence critical clearing time is basic to increasing critical clearing time when transient stability is a limiting condition. Moreover, there continues to be interest in developing approximate methods for evaluating transient stability [3], [12], [13]. While we do not address these approximate methods in this paper, we note that our exact sensitivity calculation can be used to test and validate any sensitivities that could be obtained via the approximations. The first-order sensitivity of the critical clearing time is also a useful linearization for probabilistic approaches to transient stability [20].

APPENDIX: ADJUSTMENTS NEEDED FOR DIFFERENTIAL-ALGEBRAIC MODELS

The main text writes the power system model as differential equations (1) and (3) for simplicity of expression, whereas the power system model is often differential-algebraic. This appendix summarizes the necessary adjustments for the fault-on model [18].

Suppose that the fault-on power system differential-algebraic equations are in the semi-implicit form

$$\dot{x} = g(x, y, \alpha) \tag{58}$$
$$0 = h(x, y, \alpha) \tag{59}$$

where $y \in \mathbb{R}^m$ is the algebraic state and g and h are smooth functions. Write $z = (x, y)$ and write the solution of (58), (59) with initial condition $z_0 = (x_0, y_0)$ as

$$\phi^{\mathrm{DA}}(z_0, t, \alpha) = (\phi^{\mathrm{D}}(z_0, t, \alpha), \phi^{\mathrm{A}}(z_0, t, \alpha)) \tag{60}$$

where D indicates differential and A indicates algebraic.

We assume that we are working in an open set in which we can (in principle and not usually explicitly) solve (59) to obtain $y = k(x, \alpha)$, and that h_y is nonsingular to ensure index one. (Also, the MATLAB numerical integration routines that we use require index one.) Then the fault-on differential equations equivalent to (58) and (59) are given by

$$\dot{x} = g(x, k(x, \alpha), \alpha) = f(x, \alpha) \tag{61}$$

which is identical to (1). The derivation for the fault-on power system then proceeds exactly as in the main text with the differential equations (61). However, to implement numerical methods to compute the results, it is much better to work with the original differential-algebraic equations (58), (59).

Noting that (15) becomes

$$\phi_t^{\mathrm{D}}(z^s(\alpha), t, \alpha) = g(\phi^{\mathrm{D}}(z^s(\alpha), t, \alpha), \phi^{\mathrm{A}}(z^s(\alpha), t, \alpha), \alpha) \quad (62)$$

$$0 = h(\phi^{\mathrm{D}}(z^s(\alpha), t, \alpha), \phi^{\mathrm{A}}(z^s(\alpha), t, \alpha), \alpha), \quad (63)$$

the variational equations (17) and (19) along the base case fault-on trajectory become

$$\phi_{zt}^{\mathrm{D}}|_{(z_0^s, t, \alpha_0)} = g_x \phi_z^{\mathrm{D}}|_{(z_0^s, t, \alpha_0)} + g_y \phi_z^{\mathrm{A}}|_{(z_0^s, t, \alpha_0)} \quad (64)$$

$$0 = h_x \phi_z^{\mathrm{D}}|_{(z_0^s, t, \alpha_0)} + h_y \phi_z^{\mathrm{A}}|_{(z_0^s, t, \alpha_0)} \quad (65)$$

$$\phi_{\alpha t}^{\mathrm{D}}|_{(z_0^s, t, \alpha_0)} = g_x(\phi_z^{\mathrm{D}}|_{(z_0^s, t, \alpha_0)} z_\alpha^s|_{\alpha_0} + \phi_\alpha^{\mathrm{D}}|_{(z_0^s, t, \alpha_0)})$$
$$+ g_y(\phi_z^{\mathrm{A}}|_{(z_0^s, t, \alpha_0)} z_\alpha^s|_{\alpha_0} + \phi_\alpha^{\mathrm{A}}|_{(z_0^s, t, \alpha_0)}) + g_\alpha \quad (66)$$

$$0 = h_x(\phi_z^{\mathrm{D}}|_{(z_0^s, t, \alpha_0)} z_\alpha^s|_{\alpha_0} + \phi_\alpha^{\mathrm{D}}|_{(z_0^s, t, \alpha_0)})$$
$$+ h_y(\phi_z^{\mathrm{A}}|_{(z_0^s, t, \alpha_0)} z_\alpha^s|_{\alpha_0} + \phi_\alpha^{\mathrm{A}}|_{(z_0^s, t, \alpha_0)}) + g_\alpha, \quad (67)$$

where $g_x, g_y, h_x, h_y, g_\alpha, h_\alpha$ in (64), (65), (66), (67) are evaluated at $(\phi^{\mathrm{DA}}(z_0^s, t, \alpha_0), \alpha_0)$.

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