It can be seen that the Vinnakota and Rao converters using cascade subtractors exhibit more delay than the Andraros and Ahmad design. Note that in the Andraros and Ahmad design, the full adders needed can be reduced [10] by noting that one of the inputs B in (4) is a constant 1 or 0 for n LSB's, which we have not considered in the above evaluation in (5a). The high-speed version of the Vinnakota and Rao converter exhibits similar delay and area requirements as the Andraros and Ahmad design. However, both the Piestrak's cost-effective and high-speed designs are superior to Vinnakota and Rao's high-speed as well as cost-effective designs.

IV. Conclusion

It has been shown that a recently described RNS-to-binary conversion technique for the moduli set $2^n - 1, 2^n$, and $2^n + 1$ is a variation of the well-known MRC technique. An evaluation of the area and delay performance of this technique and comparison to the Andraros and Ahmad technique described in this brief has shown that the high-speed version of Vinnakota and Rao's converter is comparable in performance to the Andraros and Ahmad technique. However, both the cost-effective and high-speed designs of Piestrak, which are an improvement over the Andraros and Ahmad technique, are shown to be superior to Vinnakota and Rao's converter, regarding area as well as conversion delay.

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Justification of Torque per Unit Velocity Methods of Analyzing Subsynchronous Resonance and a Swing Mode in Power Systems

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***Abstract*— Torque per unit velocity is a practical method to study the subsynchronous resonance instability of electric power systems. The torque per unit velocity method is justified as an eigenvalue perturbation technique and extended to power systems with thyristor switching devices and multiple torsional modes. A new method to estimate the damping of the swing mode is proposed.**

I. Introduction

Subsynchronous resonance (SSR) is an electromechanical power system instability in which transmission lines compensated with series capacitors interact with torsional modes of generator shafts [2], [10]. This instability can break generator shafts and must be studied and prevented when series compensation is used.

The SSR modes are torsional modes of the generator rotor which can be destabilized by interaction with the electrical system. The torque per unit velocity method, due to Bowler, Hedin, and others [3], [4], [9], [14], is commonly used to estimate damping of SSR modes. Its basic idea is to trace the effect of a small sinusoidal mechanical disturbance in the generator rotor velocity through the electrical network. This mechanical displacement causes a displacement of the electromagnetic torque of the generator which acts to damp or to undamp the initial displacement. The damping of an SSR mode is then estimated from the ratio of the electromagnetic torque to the generator rotor velocity. This approximation has been established for time invariant cases such as when the transmission line is compensated with a fixed series capacitor and the three ac phases are balanced.

This paper mathematically justifies the torque per unit velocity method in a general power network which includes multiple machines and thyristor switching devices. The torque per unit velocity method is derived as an eigenvalue perturbation technique which relies on the smallness of the rotor electromagnetic acceleration.

One reason to extend the analysis to account for thyristor switchings arises from the emerging technology of flexible ac transmission (FACTS). Flexible ac transmission systems such as the thyristor controlled series capacitor offer the possibility of power flow control together with suppression of SSR instabilities [5], [15], [16]. The thyristor switchings make the system equations time varying by switching so that the 3 phases of the power system are unbalanced for some portions of the supply cycle. Time variation due to phase unbalance cannot be removed by Park's transformation.

One advantage of clarifying the mathematical basis of the torque per unit velocity method is that useful extensions can be developed. We extend the method to the multiple torsional modes which occur when two generators have the same rotor frequencies. We also develop a new method to estimate the damping of the low-frequency power swing mode in the single machine case. The methods justified

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here are informally explained, applied, and validated on a test case in the companion paper [18].

II. Linearized System Equations

This section presents linearized electromechanical equations of a general multimachine power system which includes thyristor switching devices. The detailed equations for these systems are complicated, and this paper requires only their general form. Detailed linearized electromechanical equations for balanced multimachine power systems without thyristor switching devices are derived in [1] and [2]. Modifications to incorporate thyristor switching devices are detailed in [12], [17], [19].

The mechanical system consists of the turbine shafts of all the generators and is modeled as a linear, torsional spring, lumped mass system with matrix A_m. The mechanical state x_m describes the shaft angles and velocities, and its linearized deviation about the steady-state is denoted by Δx_m. The linearized mechanical system equations can be written as

$$\Delta \dot{x}_m = A_m \Delta x_m + \epsilon A_{me}(t) \Delta x_e. \tag{2.M}$$

The term $\epsilon A_{me}(t)\Delta x_e(t)$ represents the linearized coupling from the electrical system to the mechanical system via the electromagnetic machine torques. The linearized deviation about the steady state of the electrical system is denoted by Δx_e. ϵ is a small number.

The electrical system includes the electrical parts of the system and its controls and has state x_e (e.g., [17]). When thyristor switching devices are included, the electrical system equations change structure at each thyristor switching. In particular, x_e includes the reactor current of a thyristor controlled reactor only when the thyristor is conducting [12], [17]. The switching instants are determined either by the thyristor current becoming zero or by control of the thyristor firing angle. The firing angle control can depend in a complicated way on previous system states. For examples, see [12]. The linearized electrical equations have the overall form

$$\Delta \dot{x}_e = f(\Delta x_e, \Delta x_m, t). \tag{2.E}$$

Equation (2.E) is time varying, and varies in structure at each thyristor switching.

The assumptions necessary in the sequel are:

Assumption 1: The nonlinear electromechanical system equations have a periodic steady state (periodic orbit) of period T_0.

Assumption 2: Near the periodic steady state, the nonlinear electromechanical system equations are smooth between switchings, and each switching time depends smoothly on the system states at or before the switching. (In particular, there are no switching time bifurcations [13].) Smooth dependence of the equations and switching times on the parameter ϵ for small ϵ is also assumed.

Assumptions 1 and 2 imply that the electromechanical system equations can be linearized about the periodic steady state to produce the time varying linear equations (2.M), (2.E) which are periodic with period T_0. In particular, $A_{me}(t)$ and f are periodic in time with period T_0.

Assumption 3: The generator shafts are modeled as a linear, torsional spring, lumped mass system [2], and the mechanical dampings of the shafts are zero. (It is usual in SSR analysis to add the small natural mechanical dampings to modal dampings computed with the assumption of zero mechanical damping.) It follows that the matrix A_m can be transformed by a change of coordinates to diag$\{D_1, D_2, \cdots, D_n\}$, where $D_i = \begin{pmatrix} 0 & 1 \\ -\omega_i^2 & 0 \end{pmatrix}$ and ω_i is the natural frequency of the ith torsional mode. Thus, all eigenvalues of A_m have the form $\pm j\omega_i$.

Assumption 4: The rotor electromagnetic acceleration term $\epsilon A_{me}(t)\Delta x_e$ in (2.M) is small so that ϵ is a small number. ($\epsilon A_{me}(t)\Delta x_e$ is typically 2 orders of magnitude smaller than other terms in [1].)

When $\epsilon = 0$, there is no coupling from the electrical equations to the mechanical equation (2.M), and the eigenvalues $\pm j\omega_i$ of A_m are eigenvalues of the electromechanical system. The SSR and swing modes of the electromechanical system are most easily described when $\epsilon = 0$: the SSR modes correspond to the eigenvalues $\pm j\omega_i$ with $\omega_i \neq 0$ and swing modes correspond to eigenvalues $\pm j\omega_i$ with $\omega_i = 0$.

Assumption 5: System modes which are not SSR or swing modes are asymptotically stable and well damped.

The objective of the torque per unit velocity method is to estimate the damping of a particular SSR mode. The main idea is to solve the electromechanical equations (2.M), (2.E) for $\epsilon = 0$ and then to use this solution to estimate the change in the eigenvalue of the SSR mode when ϵ is nonzero.

When $\epsilon = 0$, the mechanical equation (2.M) is decoupled from the electrical equation (2.E) and is easily solved for the SSR mode. If the SSR mode is the ith mode then the corresponding solution to (2.M) is $\Delta x_m(t) = e^{j\omega_i t} v_i$, where ω_i is the frequency of mode i and v_i is the corresponding mode shape or right eigenvector scaled as described in Appendix A. Substituting $\Delta x_m(t) = e^{j\omega_i t} v_i$ in the electrical equation (2.E) yields an electrical differential equation whose steady-state solution $\Delta x_e(t)$ is the electrical part of the solution for $\epsilon = 0$. This steady-state electrical solution will cause a torque ΔT_i which acts on mode i. (ΔT_i is precisely defined and shown to be proportional to components of the term $\epsilon A_{me}(t)\Delta x_e$ in (2.M) in (A1), (A2) in Appendix A.) To summarize, ΔT_i is the steady state torque response in mode i when the mechanical input to the linearized electrical equation (2.E) is $\Delta x_m(t) = e^{j\omega_i t} v_i$, where v_i is the right eigenvector of A_m corresponding to eigenvalue $j\omega_i$.

It is proved below that ΔT_i can be used to estimate the damping of the SSR mode to order ϵ. Since (2.E) is time varying with period T_0, the response ΔT_i contains other frequency components as well as a component $c_i e^{j\omega_i t}$ of frequency ω_i. That is, it follows from the theory of periodically varying linear systems [6] that

$$\Delta T_i = c_i e^{j\omega_i t} + e^{j\omega_i t} \sum_{k \neq 0} d_k e^{j2k\pi/T_0}. \tag{2.1}$$

c_i specifies the component of ΔT_i at the subsynchronous frequency ω_i of the mechanical input.

Some means of computing ΔT_i is required for the torque per unit velocity method and time domain simulation is one practical approach. A detailed time domain simulation of the electrical system is often available because of its use in evaluating the transient torques caused by large signal deviations and in other system assessments. The steady-state simulation can be perturbed by a small sinusoidal mechanical input and the deviation in torque about the steady-state can be measured. Then the steady-state torque response of the electrical system to the complex signal $e^{j\omega_i t}$ can be evaluated as (response to $\cos\omega_i t) + j$ (response to $\sin\omega_i t$). (Since the electrical system is time varying, the response to $\sin\omega_i t$ is not simply a time shift of the response to $\cos\omega_i t$.) A change to modal coordinates then yields ΔT_i. One advantage of the method is that even when the entire electromechanical system is unstable, the electrical part of the system used to compute the electrical torque response is typically stable (see Assumption 5). Thus, the practical difficulties of simulating an unstable system are avoided. The solutions to (2.E) can also be computed directly by harmonic admittance [11] or other methods.

According to the theory of periodically varying linear systems [6], the general modal solution of (2.M), (2.E) may be expressed as

$$\begin{pmatrix}\Delta x_m(t)\\ \Delta x_e(t)\end{pmatrix} = e^{\lambda t}\begin{pmatrix}g_m(t)\\ g_e(t)\end{pmatrix} = e^{\lambda t} g(t) \tag{2.2}$$

where λ is the modal eigenvalue and the mode shape $g(t)$ is periodic in time so that $g(t) = g(t+T_0)$. Substituting $\Delta x_m(t) = e^{\lambda t} g_m(t)$ into (2.M) yields

$$(A_m - \lambda I) g_m(t) = -\epsilon A_{me}(t) g_e(t) + \dot{g}_m(t). \tag{2.3}$$

Write overbar for average over one period so that

$$\overline{h(t)} = \frac{1}{T_0}\int_0^{T_0} h(t)\,dt.$$

Since $\overline{\dot{g}_m(t)} = (1/T_0)(g_m(T_0) - g_m(0)) = 0$, averaging (2.3) gives

$$(A_m - \lambda I)\overline{g_m(t)} = -\overline{\epsilon A_{me}(t) g_e(t)}. \tag{2.4}$$

Equation (2.4) is an intermediate result useful in Sections III and IV.

III. Damping of SSR Modes

This section estimates the damping of an SSR mode with natural frequency $\omega_i \neq 0$ distinct from other mechanical frequencies. The generalization to the case of multiple SSR modes with the same frequency is given in Appendix C.

In the case of an SSR mode with a frequency $\omega_i \neq 0$ distinct from other mechanical frequencies, Assumption 3 implies that $j\omega_i$ is a simple eigenvalue of A_m with corresponding left and right eigenvectors u_i and v_i. u_i and v_i are scaled as described in Appendix A and satisfy $u_i v_i = 1$. When $\epsilon = 0$, this mode is uncoupled from the electrical system and the modal solution is $e^{j\omega_i t}\binom{v_i}{g_e^0(t)}$, where $\Delta x_e(t) = e^{j\omega_i t} g_e^0(t)$ is the solution to (2.E) when the mechanical input $\Delta x_m(t) = e^{j\omega_i t} v_i$.

When $\epsilon \neq 0$, the modal eigenvalue changes to $j\omega_i + \gamma_i$ and the modal solution changes to $e^{(j\omega_i+\gamma_i)t}\binom{g_m(t)}{g_e(t)}$. Appendix B shows that these changes are of order ϵ:

$$\begin{pmatrix}g_m(t)\\ g_e(t)\end{pmatrix} - \begin{pmatrix}v_i\\ g_e^0(t)\end{pmatrix} = O(\epsilon), \qquad \gamma_i = O(\epsilon). \tag{3.1}$$

It follows by substituting $j\omega_i + \gamma_i$ for λ in (2.4) that

$$\gamma_i \overline{g_m(t)} = \overline{\epsilon A_{me}(t) g_e(t)} + (A_m - j\omega_i I)\overline{g_m(t)}. \tag{3.2}$$

Premultiplying (3.2) on the left by u_i and using $u_i(A_m - j\omega_i I) = 0$, we get

$$u_i \gamma_i \overline{g_m(t)} = u_i \overline{\epsilon A_{me}(t) g_e(t)}. \tag{3.3}$$

Using (3.1), $u_i v_i = 1$ and (3.3), the modal eigenvalue perturbation γ_i satisfies

$$\gamma_i = u_i \overline{\epsilon A_{me}(t) g_e^0(t)} + O(\epsilon^2). \tag{3.4}$$

It remains to express $u_i \overline{\epsilon A_{me}(t) g_e^0(t)}$ in terms of the modal torque ΔT_i. Appendix A proves that

$$u_i \epsilon A_{me}(t) \Delta x_e(t) = -\Delta T_i/(2j\omega_i M_i) \tag{3.5}$$

where M_i is the modal inertia. It follows from (3.5) and $\Delta x_e(t) = e^{j\omega_i t} g_e^0(t)$ that

$$u_i \overline{\epsilon A_{me}(t) g_e^0(t)} = -\overline{e^{-j\omega_i t}\Delta T_i}/(2j\omega_i M_i).$$

But consideration of the form of ΔT_i in (2.1) shows that $\overline{e^{-j\omega_i t}\Delta T_i} = (1/T_0)\int_0^{T_0} e^{-j\omega_i t}\Delta T_i\,dt = c_i$. Thus, (3.4) becomes

$$\gamma_i = -c_i/(2j\omega_i M_i) + O(\epsilon^2). \tag{3.6}$$

The modal damping is the real part of $-(j\omega_i + \gamma_i)$:

$$\text{modal damping} = \text{Real}\left\{\frac{c_i}{2j\omega_i M_i}\right\} + O(\epsilon^2). \tag{3.7}$$

Note that the modal damping is independent of the other frequency components of ΔT_i in (2.1) to $O(\epsilon)$.

IV. Damping and Frequency of Swing Mode

This section derives a new torque per unit velocity method to estimate the frequency of the swing mode in the single machine case.

In the single machine case, it follows from Assumption 3 that A_m has a unique nontrivial Jordan block with two zero eigenvalues which corresponds to the swing mode. Thus, the two zero eigenvalues of A_m have only one left eigenvector u_i and only one right eigenvector v_i and $u_i v_i = 0$. Let u_{i+1} be the generalized left eigenvector corresponding to the double zero eigenvalue so that $u_{i+1}A_m = u_i$ and $u_{i+1}v_i = 1$. Appendix A specifies these eigenvectors in more detail.

When $\epsilon = 0$, the modal solution is $\binom{v_i}{g_e^0(t)}$, where $g_e^0(t)$ is the solution to (2.E) when the mechanical input $\Delta x_m(t) = v_i$.

When $\epsilon \neq 0$, the modal eigenvalues change to γ_i, γ_{i+1} and the modal solutions change to $e^{\gamma_j t}\binom{g_{mj}(t)}{g_{ej}(t)}$, $j = i, i+1$. Appendix B shows that because of the nontrivial Jordan form, these changes are of order $\epsilon^{1/2}$:

$$\begin{pmatrix}g_{mj}(t)\\ g_{ej}(t)\end{pmatrix} - \begin{pmatrix}v_i\\ g_e^0(t)\end{pmatrix} = O(\epsilon^{1/2}), \qquad \gamma_j = O(\epsilon^{1/2}). \tag{4.1}$$

Premultiply (2.4) by u_i for $j = i, i+1$ to obtain

$$\gamma_j u_i \overline{g_{mj}(t)} = u_i \overline{\epsilon A_{me}(t) g_{ej}(t)}. \tag{4.2}$$

Premultiply (2.4) by $u_{i+1}(A_m - \gamma_j I)$; use

$$u_{i+1}(A_m - \gamma_j I)^2 = -2\gamma_j u_i + \gamma_j^2 u_{i+1}$$

(4.2) and (A4) from Appendix A to get

$$\gamma_j^2 u_{i+1}\overline{g_{mj}(t)} = u_i \overline{\epsilon A_{me}(t) g_{ej}(t)}. \tag{4.3}$$

Now use (4.1) and (A3) to obtain for $j = i, i+1$:

$$\gamma_j^2 = u_i \overline{\epsilon A_{me}(t) g_e^0(t)} + O(\epsilon^{3/2}) = \frac{-c_i}{M_i} + O(\epsilon^{3/2}) \tag{4.4}$$

where c_i is the constant (zero frequency) component of ΔT_i. In (4.4), observe that $g_e^0(t)$, the solution of (2.E) with $\Delta x_m(t) = v_i$, is real and therefore c_i is real. Hence, the estimate of γ_j^2 is real and the estimate of γ_j is either purely imaginary or purely real. For a stable swing mode, the estimate of γ_j is purely imaginary so that $\gamma_j = \pm j\beta_j$ and the estimated modal damping is zero.

We suggest the following iterative method to better estimate both the damping and frequency of this mode. Write $\gamma = \alpha + j\beta$ and let $e^{\gamma t} g_e(t)$ be the solution of (2.E) with $\Delta x_m(t) = e^{\gamma t} v_i$ and let $e^{j\beta t} g_{e\beta}(t)$ be the solution of (2.E) with $\Delta x_m(t) = e^{j\beta t} v_i$. For practical power systems $\gamma \approx j\beta$, so $g_e(t) \approx g_{e\beta}(t)$. Thus, it is reasonable to approximate the modal solution by $e^{\gamma t}\binom{v_i}{g_{e\beta}(t)}$. Now, working as above and after some algebra, we obtain the following analog of (4.4):

$$\gamma^2 \approx -c_i(j\beta)/M_i \tag{4.5}$$

where $c_i(j\beta)$ specifies the frequency β component of ΔT_i so that $\Delta T_i = c_i(\beta j)e^{j\beta t}+$ other frequencies. The solution of this equation is estimated by the iteration: $\gamma^{[0]} = 0$, $\gamma^{[k]} = \alpha^{[k]} + j\beta^{[k]}$ and

$$\gamma^{[k+1]} = \pm h\left(\gamma^{[k]}\right) = \pm\sqrt{-c_i(j\beta^{[k]})/M_i} \tag{4.6}$$

where $c_i(j\beta^{[k]})$ specifies the $\beta^{[k]}$ frequency component of ΔT_i in response to a mechanical input $\Delta x_m(t) = e^{j\beta^{[k]}t} v_i$. According to the contraction mapping theorem, this iteration will converge for sufficiently small ϵ, because then h has Lipschitz constant less than one near the origin [note that $c_i(j\beta^{[k]})/M_i = O(\epsilon)$].

V. Conclusion

This paper derives the torque per unit velocity method for estimating the damping of SSR swing modes as an eigenvalue perturbation method. The derivation clarifies the mathematical basis of the torque per unit velocity method and extends its validity to multimachine power systems with thyristor switching devices and unbalanced phases and to the case of multiple modes with a common frequency.

An iterative torque per unit velocity method for estimating the damping and frequency of the swing mode is derived in the case of a single machine.

The method is easier than exact eigenanalysis of the entire electromechanical system. In particular, the method can estimate eigenvalues by computing the torque response of a time domain simulation of the electrical system to a small sinusoidal disturbance to the steady state. Testing of the torque per unit velocity method in [18] on the IEEE SSR first benchmark model with a thyristor controlled series capacitor shows excellent agreement with exact eigenanalysis [17].

Appendix A
Electromagnetic Acceleration

Eigenvector scalings are defined, and an expression for the linearized electromagnetic acceleration is derived. Choose $x_m = (\theta_1, \cdots, \theta_n, \dot{\theta}_1, \cdots, \dot{\theta}_n)^t$, where θ_i and $\dot{\theta}_i$ are the angle and velocity of shaft i. Then $A_m = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -J^{-1}K & 0_{n\times n} \end{pmatrix}$, where $J = \operatorname{diag}\{J_1, \cdots, J_n\}$ is the inertia matrix and K is the spring constant matrix. Now the acceleration

$$\epsilon A_{me}(t)\Delta x_e(t) = \begin{pmatrix} 0_n \\ -J^{-1}\underline{\Delta T} \end{pmatrix}$$

where $\underline{\Delta T} \in \mathbb{R}^n$ is the vector of linearized electromagnetic torques on the shaft [2]. Let the columns of the $n \times n$ matrix $Q = (q_1\, q_2 \cdots q_n)$ represent the mode shapes of the shaft angles. That is, q_i is the ith right eigenvector of $J^{-1}K$. Let $P = (p_1^t \cdots p_n^t)^t = Q^{-1}$ so that p_i is the ith left eigenvector of $J^{-1}K$. The ith modal torque is defined as

$$\Delta T_i = p_i \underline{\Delta T}. \tag{A1}$$

For the SSR mode number i with $\omega_i \neq 0$, choose $v_i = (q_i^t,\, j\omega_i q_i^t)^t$ and $u_i = \frac{1}{2}(p_i,\, p_i/(j\omega_i))$. Then

$$\begin{aligned} u_i \epsilon A_{me}(t)\Delta x_e(t) &= -(2j\omega_i)^{-1} p_i J^{-1}\underline{\Delta T} \\ &= -\Delta T_i/(2j\omega_i M_i) \end{aligned} \tag{A2}$$

where $M = PJQ = \operatorname{diag}\{M_1, \cdots, M_n\}$ is the modal inertia matrix. Writing e_i for a row vector with all zeros except for 1 in the ith place, the detailed steps to obtain (A2) are $p_i J^{-1} = e_i P J^{-1} Q P = e_i M^{-1} P = M_i^{-1} e_i P = M_i^{-1} p_i$ and using (A1).

For the swing mode case with $\omega_i = \omega_{i+1} = 0$, choose $v_i = (q_i,\, 0)^t$, $u_i = (0,\, p_i)$ and $u_{i+1} = (p_i,\, 0)$. Then

$$u_i \epsilon A_{me}(t)\Delta x_e(t) = \Delta T_i/M_i \tag{A3}$$

and

$$u_{i+1} A_{me}(t)\Delta x_e(t) = (p_i,\, 0_n)(0_n,\, J^{-1}\underline{\Delta T})^t = 0. \tag{A4}$$

Appendix B
Eigenvalue Perturbation Estimates

This appendix justifies the estimates of the variation in eigenvalues used in Sections II and IV. References [7], [8], and [20] are useful background.

Write J^ϵ for the Jacobian of the Poincaré map evaluated at the periodic orbit. It follows from Assumption 2 that J^ϵ exists and is a smooth function of ϵ for small ϵ. Let z^ϵ be a right eigenvector of J^ϵ corresponding to the nonzero eigenvalue $e^{\gamma^\epsilon T}$ of J^ϵ. Let $p^\epsilon(t)$ be the solution of the linearized equations (2.M), (2.E) with initial condition $p^\epsilon(0) = z^\epsilon$. We have $p^\epsilon(T) = J^\epsilon p^\epsilon(0) = e^{\gamma^\epsilon T} p^\epsilon(0)$, and flowing forward for time t yields $p^\epsilon(t+T) = e^{\gamma^\epsilon T} p^\epsilon(t)$. Defining $g^\epsilon(t) = e^{-\gamma^\epsilon t} p^\epsilon(t)$, it follows that $g^\epsilon(t) = g^\epsilon(t+T)$ with $g^\epsilon(0) = p^\epsilon(0) = z^\epsilon$. Thus $p^\epsilon(t) = e^{\gamma^\epsilon t} g^\epsilon(t)$ is the solution of the mode with eigenvalue γ^ϵ and whose mode shape is the periodic function $g^\epsilon(t)$.

Since J^ϵ is a smooth function of ϵ, it follows from [20] that if $e^{\gamma^0 T}$ is a simple eigenvalue of J^0 with right eigenvector z^0, then J^ϵ has an eigenvalue $e^{\gamma^\epsilon T}$ with right eigenvector z^ϵ that satisfies $e^{\gamma^\epsilon T} - e^{\gamma^0 T} = O(\epsilon)$ and $z^\epsilon - z^0 = O(\epsilon)$. The analysis when J^0 has a multiple eigenvalue with simple elementary divisors is similar. However, when J^0 has a double eigenvalue at $e^{\gamma^0 T}$ with only one corresponding independent eigenvector z^0, then [20] shows that J^ϵ has two eigenvalues $e^{\gamma_1^\epsilon T}$ and $e^{\gamma_2^\epsilon T}$ with right eigenvectors z_1^ϵ and z_2^ϵ such that $e^{\gamma_j^\epsilon T} - e^{\gamma_j^0 T} = O(\epsilon^{1/2})$ and $z_j^\epsilon - z^0 = O(\epsilon^{1/2})$ for $j = 1, 2$.

Assumption 5 implies that the eigenvalues of J^0 corresponding to the SSR and swing modes are distinct from other, well damped system eigenvalues. Therefore, if γ^0 is a simple eigenvalue of A_m, $e^{\gamma^0 T}$ is a simple eigenvalue of J^0, and therefore perturbations of this eigenvalue and corresponding eigenvectors are $O(\epsilon)$. Similarly, if $\gamma^0 = 0$ is a double eigenvalue of A_m with only one eigenvector, then J^0 has a double eigenvalue at $e^{0T} = 1$ and perturbations of these eigenvalues and corresponding eigenvectors are $O(\epsilon^{1/2})$.

Now $z_i^\epsilon - z^0 = p^\epsilon(0) - p^0(0) = O(\epsilon)$ implies that $p^\epsilon(t) - p^0(t) = O(\epsilon)$ for $0 \le t \le T$, and hence the estimate (3.1) holds. Estimate (4.1) follows similarly.

Appendix C
Multiple SSR Modes

We show how to estimate the damping of multiple SSR modes having the same nonzero natural frequency. We assume that, say, the first two torsional modes have the same nonzero natural frequency; the analysis for more than two such modes is similar. Then A_m has two eigenvalues $j\omega_1 = j\omega_2$ with corresponding left and right real eigenvectors u_1, u_2, and v_1, v_2 with $u_1 v_2 = u_2 v_1 = 0$.

When $\epsilon = 0$, the general modal solution is

$$a_1 e^{j\omega_1 t}\begin{pmatrix} v_1 \\ g_{e1}^0(t) \end{pmatrix} + a_2 e^{j\omega_2 t}\begin{pmatrix} v_2 \\ g_{e2}^0(t) \end{pmatrix}$$

where a_1, a_2 are (as yet) undetermined constants.

When $\epsilon \neq 0$, the modal eigenvalues change by $O(\epsilon)$ to $j\omega_1 + \gamma_1$ and $j\omega_2 + \gamma_2$ and the modal solution changes by $O(\epsilon)$ to

$$a_1 e^{(j\omega_1+\gamma_1)t}\begin{pmatrix} g_{m1}(t) \\ g_{e1}(t) \end{pmatrix} + a_2 e^{(j\omega_2+\gamma_2)t}\begin{pmatrix} g_{m2}(t) \\ g_{e2}(t) \end{pmatrix}. \tag{C1}$$

(The values of a_1 and a_2 are assumed constant in order to conclude that the modal solution changes by $O(\epsilon)$ [20].)

Premultiplying (3.2) by u_1 and u_2 yields for $i = 1, 2$

$$\begin{aligned} u_1 \gamma_i \overline{g_m(t)} &= u_1 \overline{\epsilon A_{me}(t) g_e(t)} \\ u_2 \gamma_i \overline{g_m(t)} &= u_2 \overline{\epsilon A_{me}(t) g_e(t)}. \end{aligned}$$

Substituting the modal solution (C1) as in Section III and neglecting terms of $O(\epsilon^2)$ yields

$$\gamma_i a_1 = u_1 \overline{\epsilon A_{me}(t) g_{e1}^0(t)}\, a_1 + u_1 \overline{\epsilon A_{me}(t) g_{e2}^0(t)}\, a_2$$
$$\gamma_i a_2 = u_2 \overline{\epsilon A_{me}(t) g_{e1}^0(t)}\, a_1 + u_2 \overline{\epsilon A_{me}(t) g_{e2}^0(t)}\, a_2$$

or

$$\gamma_i \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -B \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

where

$$B = \frac{1}{2j} \begin{pmatrix} c_1(v_1)/(M_1\omega_1) & c_1(v_2)/(M_1\omega_1) \\ c_2(v_1)/(M_2\omega_2) & c_2(v_2)/(M_2\omega_2) \end{pmatrix}.$$

The notation $c_i(v_k)$ denotes the $e^{j\omega_i t}$ coefficient of ΔT_i in response to a mechanical displacement $\Delta x_m = e^{j\omega_k t} v_k$.

For a nontrivial solution, a_1 and a_2 cannot both be zero, and $-\gamma_1$ and $-\gamma_2$ are the eigenvalues of B (the eigenvectors of B determine a_1 and a_2). Hence, the modal dampings are the real parts of the eigenvalues of B to $O(\epsilon^2)$.

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