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Instability Tests, Lyapunov's Direct Method, and Exact Stability Boundaries for Flexible Satellites

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Simple tests are developed to determine instability regions in the parameter space of flexible satellites. These tests are used in conjunction with Lyapunov's direct method to obtain exact stability boundaries in parameter space in cases where Lyapunov methods give only sufficient conditions for stability. In the case of Hamiltonian systems, the tests reduce to checking the sign of the determinant of the Hessian of the Hamiltonian at a single point. The Hamiltonian assumption can be relaxed to allow the inclusion of damping terms provided these terms have no linear dependence on position. Previous studies use complete or pervasive damping assumptions to obtain exact stability boundaries, whereas the proposed methods work for no damping or the damping described earlier. Examples illustrate the usefulness of the tests.

Introduction

FOR the past 30 years there has been considerable interest in characterizing the stability of satellites in terms of system parameters. It is well known that sufficient conditions for stability of systems in terms of parameters can often be obtained by using Lyapunov's direct method with the system Hamiltonian as a testing function, e.g., Refs. 1–22. Since Lyapunov methods often only give sufficient conditions for stability, the exactness of the stability boundary obtained is sometimes unclear. This paper provides a simple test for proving that the region on the other side of the stability boundary is unstable and hence that the stability boundary is exact. A key feature of the test is that instability of an entire region of parameter space may be deduced by testing a single point in the region.

The first major paper applying Lyapunov's direct method to determine stability conditions for satellites with moving parts was that of Pringle.[21] This paper and subsequent papers in the 1970s used the assumption of complete or pervasive damping to determine exact stability boundaries with Lyapunov methods (e.g., Refs. 1, 3, 10, 15–17, 19, 22, and 23). More recently, Walker[24] used pervasive damping type assumptions to generate methods for determining exact stability boundaries of generalized equilibria for pseudodissipative systems [see conditions 3.1(d) and 3.2(e) of Ref. 24]. When our tests apply, they yield exact stability boundaries for Hamiltonian systems that possess no damping and for Hamiltonian systems with additional damping terms provided these terms have no linear dependence on position.

This paper considers the stability of an equilibrium at the origin and presents simple tests for classifying regions of parameter space for which the origin is unstable. Tests are given for several general classes of systems and typically require the determinant of a Jacobian of the system to be computed in terms of the system parameters. The vanishing of this determinant defines boundaries of the region, and the region is confirmed to be unstable by testing the sign of the determinant at a single point within the region. The results apply to Hamiltonian systems by testing the determinant of a Hessian of the Hamiltonian in place of the determinant of a Jacobian. The results also apply if damping terms are added to the system, provided these terms have no linear dependence on position. After a few simple explanatory examples, exact stability boundaries are determined for a flexible satellite with guy wires to give a realistic example of the use of the instability tests.

The tests rely on deducing the instability of a region of parameter space from a Jacobian having an odd number of positive real eigenvalues. We do not treat the stability boundaries associated with the Hopf bifurcation and oscillatory instability in this paper. Oh (see Ref. 25, Lemma 3) has used similar hypotheses and arguments to deduce instability at a point in parameter space, and Maddocks (see Ref. 26, page 83) used these methods to prove the instability of relative equilibria in symmetric Hamiltonian systems. Our main contribution is to deduce the instability of an entire region of parameter space by testing a single point in the region and hence determine exact stability boundaries. To the best of our knowledge, this contribution is new. Practical forms of the tests in terms of determinants of submatrices of Hessians and Jacobians are then derived. In particular, a form of the test is given whereby exact stability boundaries in parameter space can be derived for a complex system after calculating the Hessian matrix of the dynamic potential of the system. This is much simpler than standard methods that require generating the equations of motion for the system, linearizing them, and computing eigenvalues. Furthermore, for high-order systems, it is often impossible to obtain a closed-form expression for the eigenvalues in terms of system parameters.

Basic Instability Lemma

Consider a system of n first-order differential equations:

$$\dot{x} = f(x, \lambda) \qquad x \in \Re^n, \quad \lambda \in \Re^m \tag{1}$$

where λ is a vector of parameters and f is a C^∞ smooth function of x and λ. We require that $f(0, \lambda) = 0$ for arbitrary λ so that the origin is always an equilibrium and we study the stability of Eq. (1) at the origin.

We write Df for the Jacobian of f evaluated at the origin and indicate the dependence of Df on the parameter vector λ by $Df(\lambda)$ when it is useful to emphasize this dependence. For convenience we say that a point λ_0 in the parameter space of system (1) has property A if the Jacobian $Df(\lambda_0)$ has an odd number of positive real eigenvalues. Since complex eigenvalues occur in complex conjugate pairs, λ_0 having property A is equivalent to $Df(\lambda_0)$ having an odd number of eigenvalues with positive real parts.

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We define Z to be the set of points in the parameter space S where Df has a zero eigenvalue; Df having a zero eigenvalue coincides with the singularity of Df and the vanishing of $\det Df$,

$$Z \equiv \{\lambda \in S \mid \det Df(\lambda) = 0\} \tag{2}$$

It is easy to see that parts of Z may bound stability regions in the parameter space. This follows from the fact that $\det Df$ passes through zero when one of the real eigenvalues of Df becomes positive.

The basic result that we prove and apply in various ways in this paper is the following lemma that shows that instability of a region of parameter space may be simply deduced if the set Z is known and property A has been confirmed at a single point of the region.

Lemma 1: Let U be a path connected region of the parameter space of system (1) that does not intersect Z. Suppose that a point $\lambda_0 \in U$ has property A. Then system (1) is unstable for all $\lambda \in U$.

Proof: Let λ_1 be any point in U and let C be a path connecting λ_0 and λ_1 that does not intersect Z. The system is unstable at λ_0 since $Df(\lambda_0)$ has an odd number of positive real eigenvalues and eigenvalues in the right half-plane imply instability. As parameters vary along C, eigenvalues cannot pass through zero because C is disjoint from Z, and if eigenvalues cross the imaginary axis, they do so in complex conjugate pairs. Therefore $Df(\lambda)$ has an odd number of eigenvalues in the right half-plane for all parameters λ on C. In particular, we conclude that the system is unstable at λ_1. □

To apply Lemma 1 directly, we need to compute $\det Df(\lambda)$ in terms of the system parameters λ to deduce the position of Z in parameter space so that U can be chosen disjoint from Z. In addition, we need to check property A at any single point λ_0 in U by computing eigenvalues of $Df(\lambda_0)$. This procedure is illustrated in Example 1 of Ref. 27. The following sections show that for many useful cases, Z and property A may be determined more simply, thus allowing instability regions to be determined with considerably less work.

Even Dimensional Systems

When applying Lemma 1 to even dimensional systems, property A can be verified by checking the sign of $\det Df$. The eigenvalues of Df need not be computed.

More precisely, λ_0 having property A is equivalent to $\det f(\lambda_0) < 0$ for even dimensional systems. This observation can be explained as follows: Since the determinant of a matrix is the product of its eigenvalues, we have that $\det Df < 0$ is equivalent to an odd number of eigenvalues of $\det Df$ in the left half-plane. (Note that the product of a complex pair of eigenvalues is positive.) Furthermore, for even dimensional systems, $\det Df \neq 0$ implies that the parity of the number of eigenvalues in the left half-plane equals the parity of the number of eigenvalues in the right half-plane. Hence $\det Df(\lambda_0) < 0$ is equivalent to an odd number of eigenvalues of $Df(\lambda_0)$ with positive real parts and λ_0 has property A.

Hamiltonian Systems

Lemma 1 can be applied to Hamiltonian systems by testing the Hessian of the Hamiltonian. The differential equations f and the Jacobian Df need not be calculated. This is useful because the Hamiltonian is often easier to derive than f.

Consider a parameterized, smooth, holonomic Hamiltonian system with Hamiltonian H and the origin as an equilibrium:

$$(\dot{q}, \dot{p}) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right) = f(q, p, \lambda) \tag{3}$$

We write D^2H for the Hessian matrix of H evaluated at the origin. The Jacobian and the Hessian are related by the identity

$$Df = JD^2H \tag{4}$$

where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

In particular,

$$\det Df = \det D^2H \tag{5}$$

since $\det J = 1$. Therefore, for Hamiltonian systems, a point is in set Z if and only if $\det D^2H = 0$. Moreover, using the fact that the dimension of a Hamiltonian system is even, we have that a point $\lambda_0 \in U$ has property A if and only if $\det Df < 0$ if and only if $\det D^2H < 0$ (also see Ref. 25). Note that only the computation of $\det D^2H$ is required to apply Lemma 1 to Hamiltonian systems. This result has been derived with the Hamiltonian expressed as a function H of the Hamiltonian coordinates q, p. It is often easier to obtain the Hamiltonian expressed as a function h of the Lagrange coordinates $q, \dot{q}$, and we now deduce the corresponding result for h.

Let the transformation between Lagrangian and Hamiltonian coordinates be given by $P = T(Q)$. Applying the chain rule for differentiation yields

$$D^2h = D(DH\,DT) = (DT)^T D^2H\,DT + DH\,D^2T \tag{6}$$

and since DH vanishes at the equilibrium,

$$D^2h = (DT)^T D^2H\,DT \tag{7}$$

Hence

$$\det D^2h = (\det DT)^2 \det D^2H \tag{8}$$

The term $(\det DT)^2$ is always positive so $\det D^2h$ may be tested in place of $\det D^2H$. That is, the set Z is given by $\det D^2h(\lambda) = 0$ and λ_0 having property A is equivalent to $\det D^2h(\lambda_0) < 0$.

Bifurcation Computations

There are standard numerical methods from bifurcation theory to compute points on a stability boundary Z, and we briefly discuss the exactness of the stability boundary and how our results relate to standard results of bifurcation theory.

In numerically computing bifurcation instabilities associated with zero eigenvalues of Df, one starts with parameter values λ_s at which the system is stable and varies λ along some curve in parameter space (usually a straight line) until stability is lost at some critical parameter value λ_* at which Df becomes singular and a bifurcation occurs. Note that λ_* is a point of Z on the boundary of a stable region. There are standard methods for computing λ_* (see Ref. 28) that can be adapted to Hamiltonian systems.[29]

The bifurcation associated with Df becoming singular is generically a transcritical bifurcation, although if the problem has symmetries, other bifurcations such as the pitchfork bifurcation can generically occur.[30] (The saddle-node bifurcation is ruled out by our assumption of an equilibrium at zero for all parameter values.) As noted earlier, we do not treat the oscillatory instability associated with the Hopf bifurcation in this paper.

For a generic bifurcation at λ_*, the zero eigenvalue is unique and simple and certain transversality conditions are satisfied. (Here generic means that an open, dense subset of systems parameterized by curves in the parameter space have these bifurcations.[30,31]) One of the transversality conditions[32] is

$$w_* D_\lambda Df(\lambda_*) v_* \neq 0 \tag{9}$$

where v_* and w_* are the right and left eigenvectors of $Df(\lambda_*)$ corresponding to the zero eigenvalue and D_λ denotes Jacobian with respect to λ. Transversality condition (9) implies that the stability boundary near λ_* is a hypersurface and that the critical eigenvalue of Df passes through zero with nonzero speed as the stability boundary is crossed. Moreover, in a small ball containing λ_*, the system is stable on one side of the stability boundary and unstable on the other side of the stability boundary.[32] Thus bifurcation theory asserts that the stability boundary is exact near λ_* for a generic bifurcation at λ_*. Moreover, since the zero eigenvalue of $Df(\lambda_*)$ is simple and unique, it is the only eigenvalue becoming positive as the stability boundary is crossed so that the unstable region sufficiently close to λ_* has Df with exactly one positive eigenvalue. That is, points in the unstable region sufficiently close to λ_* satisfy property A. It follows immediately from Lemma 1 that any region of parameter space that is bounded by the small portion of the stability boundary near λ_* and other parts of Z is unstable. We conclude that at a bifurcation with $Df(\lambda_*)$ having a unique simple zero eigenvalue and satisfying transversality condition (9), Lemma 1 extends the local exactness of the stability boundary well known in bifurcation theory to an assertion that an entire region bounded by Z is unstable.

Exact Stability Boundaries for Hamiltonian Systems

A stability boundary in the parameter space of a Hamiltonian system can sometimes be proven to be exact by using Lemma 1 to prove that the region on one side of the boundary is unstable and using Lyapunov's direct method to prove that the region on the other side of the boundary is stable. This section argues that this method of proving exactness will always work if the stability boundary is a hypersurface corresponding to a generic bifurcation in which an eigenvalue passes through zero and the stability on one side of the boundary has been established by the positive definiteness of the Hessian of the Hamiltonian. These stability boundaries are of the most practical interest.

Lyapunov's direct method can be used to find stable regions in parameter space for Hamiltonian systems by using the Hamiltonian as a testing function. In particular, the system is stable in a region of parameter space for which the Hessian D^2H of the Hamiltonian is positive definite (see Ref. 14, page 244). Sylvester's criterion (Ref. 14, page 233) implies that positive definiteness of H is equivalent to all of the principal minor determinants of D^2H being positive. Furthermore, once H is shown to be positive definite at a point in parameter space, the positive definiteness of H, and hence stability, can only be lost by $\det D^2H$ becoming zero. To see this, note that D^2H, being real, symmetric, and positive definite, has all of its eigenvalues lying on the positive, real axis. Therefore positive definiteness of D^2H cannot be lost until one of these eigenvalues passes through zero. Since Z is defined by the vanishing of $\det D^2H$, the boundary of the region guaranteed to be stable by Lyapunov's direct method is part of the set Z. As we approach a point λ_* on this boundary from the stable side, at least one of the eigenvalues of D^2H will approach the origin. If λ_* is a generic bifurcation point, exactly one eigenvalue of D^2H will pass from the right half-plane to the left half-plane as we pass through the boundary. Therefore a point near λ_* on the other side of the boundary will have property A, thus establishing the exactness of the stability boundary. In summary, a hypersurface of Z bounding a region in which $\det D^2H$ is positive is generically an exact stability boundary and its exactness may be proved using our methods (see examples 2 and 3).

Reduced-Order Method for Second-Order Systems

Since second-order differential equations have even order, the section discussing even dimensional systems shows that instability regions may be obtained by computing $\det Df(\lambda)$ to determine the set Z and checking that $\det Df(\lambda_0) < 0$ for a point λ_0 in the region. This section observes that these tests may be done for second-order systems using a submatrix of Df. Similar observations for a class of second-order systems appear in Ref. 33.

Consider the second-order system of n smooth ordinary differential equations (ODEs):

$$\ddot{q} = F(q, \dot{q}, \lambda) \tag{10}$$

If we let $y = \dot{q}$, we can write system (10) as the following system of $2n$ first-order equations:

$$(\dot{q}, \dot{y}) = [y, F(q, y, \lambda)] = f(q, y, \lambda) \tag{11}$$

and Df takes the form

$$Df = \begin{bmatrix} 0 & I \\ F_q & F_y \end{bmatrix} \tag{12}$$

Apply the Laplace expansion of the determinant[34] to obtain

$$\det Df = -\det F_q \tag{13}$$

Therefore to determine the set Z we need only compute $\det F_q$, and to check that λ_0 has property A we need only check that $\det F_q(\lambda_0) > 0$.

Reduced-Order Method for Hamiltonian Systems

The section discussing Hamiltonian systems shows that instability regions for Hamiltonian systems may be obtained by computing the determinant $\det D^2h$ of the Hessian of the Hamiltonian to determine the set Z and checking that $\det D^2h(\lambda_0) < 0$ for any point λ_0 in the region. (Recall that h is the Hamiltonian expressed in the Lagrangian coordinates $q, \dot{q}$.) This section observes that these tests may be done for the determinant of a submatrix of D^2h if the Hamiltonian has the form

$$h = h_2 + h_0 \tag{14}$$

where h_2 is a positive definite quadratic form in the generalized velocities $\dot{q}$ and h_0, called the dynamic potential, is a function only of the generalized coordinates q.

The Hessian then takes the form

$$D^2h = \begin{bmatrix} D_q^2 h_0 & 0 \\ 0 & D_{q\mathrm{dot}}^2 h_2 \end{bmatrix} \tag{15}$$

where D_q denotes derivative with respect to q and $D_{q\mathrm{dot}}$ denotes derivative with respect to $\dot{q}$ and we have

$$\det D^2h = \det D_q^2 h_0 \det D_{q\mathrm{dot}}^2 h_2 \tag{16}$$

Since h_2 is a positive definite quadratic form, $\det D_{q\mathrm{dot}}^2 h_2(\lambda) > 0$ for all λ and it follows that Z and property A may be determined from $\det D_q^2 h_0$. Also, since h_2 is a quadratic form, h is positive definite if and only if h_0 is positive definite. Thus we only need to check positive definiteness of h_0 when applying Lyapunov's direct method to this class of Hamiltonian systems (see Ref. 14, page 247).

We also note that these results apply if the Hamiltonian H is assumed to have the form

$$H = H_2 + H_0 \tag{17}$$

where H_2 is a positive definite quadratic form in the momenta p and H_0 is a function only of the generalized coordinates q.

Nonconservative Systems

This section shows that the methods for Hamiltonian systems developed in previous sections generalize to systems with additional damping terms, provided these terms have no linear dependence on position. We discuss how this result generalizes results from Lyapunov instability theory that assume complete or pervasive damping.

Consider a holonomic, conservative system with Lagrange's equations:

$$\ddot{q} = F(q, \dot{q}, \lambda) \tag{18}$$

and let Df denote the Jacobian of the state space form of Eq. (18). We also consider a holonomic nonconservative system obtained by adding a term $C(q, \dot{q}, \lambda)$ to Eq. (18) where we assume that $C_q = 0$ at the equilibrium:

$$\ddot{q} = F_{nc}(q, \dot{q}, \lambda) = F(q, \dot{q}, \lambda) + C(q, \dot{q}, \lambda) \tag{19}$$

and let Df_{nc} denote the Jacobian of the state space form of Eq. (19). Then $\det Df_{nc} = -\det (F_q + C_q) = -\det F_q$ and Eq. (13) imply that

$$\det Df_{nc} = \det Df \tag{20}$$

Equation (20) implies that property A and the set Z are independent of the nonconservative forces.

For a conservative system, $\dot{h} = 0$, whereas $\dot{h} \leq 0$ for a damped system. If $\dot{h}$ is a negative definite function of the generalized velocities $\dot{q}$, then the system is said to possess complete damping. If $\dot{h}$ is only a negative semidefinite function of the generalized velocities $\dot{q}$, but the set of points for which $\dot{h} = 0$ contains no nontrivial positive half-trajectory of the system, then the system is said to possess pervasive damping (see Ref. 14, pp. 247, 248). For Hamiltonian systems with additional complete or pervasive damping, we can use the system Hamiltonian as a testing function and deduce that the system is asymptotically stable when the Hamiltonian is positive definite. Moreover, the Lyapunov instability theorem implies that the system is unstable when the Hamiltonian is not positive definite (see Ref. 14, page 248, Ref. 21, and generalizations in Ref. 24). Thus for completely or pervasively damped Hamiltonian systems, Lyapunov methods using the Hamiltonian as a testing function give necessary and sufficient conditions for stability. For arbitrarily damped Hamiltonian systems, Lyapunov methods give only sufficient conditions for stability, and the methods of this paper are useful for determining exact stability boundaries.

Examples

Example 1

In this example, exact stability boundaries are determined with Lyapunov's direct method in the presence of nonconservative forces, even if they do not possess the form of complete or pervasive damping.

Consider a nonconservative system with kinetic energy $T = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2$, potential energy $V = \frac{1}{2}aq_1^2 + \frac{1}{2}bq_2^2$, and nonconservative forces $C = (Q_1, Q_2) = (-\beta_1\dot{q}, -\beta_2\dot{q}_2)$. The system has Hamiltonian $h = \frac{1}{2}aq_1^2 + \frac{1}{2}bq_2^2 + \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2$, where a and b are parameters. The Hamiltonian is of the form $h = h_2 + h_0$ and h_2 is a positive definite quadratic form, and so h is positive definite if and only if h_0 is positive definite. Since h_0 is positive definite in quadrant I (see Fig. 1) by Sylvester's criterion, we have from Lyapunov's direct method that the system is stable in quadrant I, provided $\dot{h} \le 0$. Expressing h as a function of q, p and noting that h does not explicitly depend on time, we have

$$\dot{h} = \dot{H} = \sum_{i=1}^{2}\left(\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i\right) = -\beta_1\dot{q}_1^2 - \beta_2\dot{q}_2^2 \qquad (21)$$

(To see this, recall that $p_i = \partial L/\partial\dot{q}_i$ where $L = T - V$. Then apply the modified Hamilton's equations as found on p. 95 of Ref. 14.) Thus, if β_1 and β_2 are greater than or equal to zero, then $\dot{h} \le 0$ and quadrant I is stable by Lyapunov's direct method. Thus, the positive a and b axes are stability boundaries when β_1 and β_2 are greater than or equal to zero. To show that they are exact stability boundaries we need to show that quadrants II and IV (see Fig. 1) are unstable.

Since the nonconservative forces have no linear dependence on position, and the Hamiltonian is of the form $h = h_2 + h_0$, we can apply the reduced-order methods for Hamiltonian systems; $D^2h_0 = \mathrm{diag}\{a, b\}$ and $\det D^2h_0 = -1 < 0$ at points $(-1, 1)$ and $(1, -1)$ in quadrants II and IV, respectively. Then Lemma 1 guarantees the instability of quadrants II and IV for arbitrary values of β_1 and β_2. Therefore the positive a and b axes are exact stability boundaries when β_1 and β_2 are greater than or equal to zero.

Instability of quadrant III cannot be shown since $\det D^2h_0 > 0$ in this quadrant. This happens because the boundary between quadrants I and III (the origin) is the intersection of two hyperplanes of Z and two eigenvalues of D^2h_0 become negative as the parameters pass from quadrant I to quadrant III at the origin. However, there is a generic transition from a stable region to an unstable region when leaving quadrant I at any point except the origin.

In the special case of β_1 and β_2, both strictly positive, the system has complete damping, and quadrants II, III, and IV are unstable by Theorem 6.9.6 (see Ref. 14, page 248), and the positive a and b axes can be proven to be exact stability boundaries by Lyapunov methods alone.

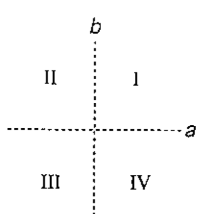

Fig. 1 Parameter space for example 1 (dashed lines = set Z).

Example 2

This example considers the attitude stability of the gravity gradient stabilized rigid satellite considered by DeBra and Delp.[35] By linearizing the equations ofmotion, DeBra and Delp derived necessary and sufficient stability conditions in parameter space that are shown in Fig. 2 (shaded regions). (The parameters are $k_1 = (C - B)/A$ and $k_2 = (A - C)/B$ where A, B, and C are the principal moments of inertia of the satellite as described in Ref. 14.) The derivations by DeBra and Delp were not rigorous, however, due to the presence of gyroscopic terms in the equations of motion.[36] Meirovitch[14] and others used Lyapunov's direct method to rigorously derive sufficient conditions for stability in region 1. We use the methods of this paper to determine unstable regions of parameter space and the exactness of the stability boundaries of region 1.

The expression for D^2h_0 as given on p. 473 of Ref. 14 is

$$D^2h_0 = \begin{bmatrix} 4\Omega^2(C - A) & 0 & 0 \\ 0 & \Omega^2(C - B) & 0 \\ 0 & 0 & -3\Omega^2(A - B) \end{bmatrix} \qquad (22)$$

Using Sylvester's criterion for positive definiteness and Lyapunov's direct method,[14] we have that sufficient conditions for stability are that $C > B > A$, which means that region 1 is stable. Setting $\det D^2H = 0$, we find that Z consists of the dashed lines shown in Fig. 2, which divide the parameter space into six regions.

Applying the techniques of this paper, we test the sign of $\det D^2h_0$ at points in regions 1–6 as summarized in Table 1. Since $\det D^2h_0 < 0$ at points tested in regions 2, 4, and 6, we find that regions 2, 4, and 6 are unstable, whereas region 1 is stable by Lyapunov's direct method. Since regions 2 and 6 are unstable, the boundaries of region 1 have been rigorously shown to be exact stability boundaries. The tests are inconclusive for regions 3 and 5. (It is known that region 5 contains both stable and unstable points.[37]) The hypersurfaces of Z (in this case curves) bounding region 1 are examples of the typical stability boundaries bounding a region in which D^2h_0 is positive definite, which can be proved to be exact.

The origin is an example of a nongeneric boundary point at which two curves of Z intersect. If the parameters pass through

Table 1 Calculation of $\det D^2h_0$ for example 2

Region	Point	A	B	C	k_1	k_2	$\det D^2h_0$
1	p1	8/9	4/3	14/9	1/4	−1/2	$(61/84)\Omega^6$
2	p2	12/5	8/5	14/5	1/2	−1/4	$(-576/125)\Omega^6$
3	p3	8	4	6	1/4	1/2	$192\Omega^6$
4	p4	24/5	4	14/5	−1/4	1/2	$(-576/25)\Omega^6$
5	p5	20/9	8/3	14/9	−1/2	1/4	$(320/81)\Omega^6$
6	p6	8/15	4/3	6/5	−1/4	−1/2	$(-64/75)\Omega^6$

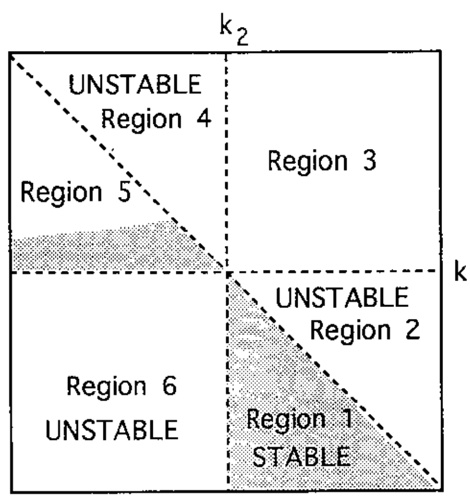

Fig. 2 Parameter space for example 2 (dashed lines = set Z).

the origin, two eigenvalues of D^2H change sign simultaneously, $\det D^2H$ remains positive, and our methods do not apply.

We note that eigenvalue analysis of the linearized equations, as done in Ref. 35, establishes the instability of region 3 and the nonshaded portion of region 5. However, this analysis requires the generation of the equations of motion and does not establish rigorously the stability of region 1. The advantage of our method is that the same computation of D^2h_0 that is required to rigorously establish stability of region 1 can be used to determine instability of regions 2 and 6, and these are the regions of interest in determining the exactness of the boundaries of region 1.

Example 3

Sufficient conditions for stability of a dual spin stabilized satellite with flexible appendages are derived in Ref. 13 via Lyapunov's direct method. Here we use the methods of this paper to construct exact stability boundaries in parameter space.

Satellite Model

The satellite considered in this paper is shown in Fig. 3 and consists of a central body containing an internal rotor and two rectangular solar panels supported by massless, elastic shafts with circular cross sections. The central body, rotor, and panels are assumed to be rigid. Each supporting shaft has polar moment of inertia J, modulus of rigidity G, and length L. A guy-wire system is assumed to constrain panel–shaft vibrations to purely torsional modes.[11–13]

The central body has principal body axes $\imath, \jmath, k$, which are also principal body axes for the entire satellite. The origin of the $\imath, \jmath, k$ coordinate system is the center of mass of the central body and the center of mass for the entire satellite. The principal moments of inertia of the panels about their principal axes are denoted A_p, B_p, and C_p and the principal axes of the panels are aligned with the axes $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ as shown in Fig. 3. The center of mass of each panel lies along the $\boldsymbol{j}$ axis and is located a distance ℓ from the center of mass of the central body. The terms α_1 and α_2 are generalized coordinates that describe the angle each panel makes with the $\jmath, k$ plane. α_1 and α_2 are also used to measure the amount of twist of each shaft. The center of mass of the rotor is located at O and the rotor spins about the $\boldsymbol{k}$ axis at a constant rate of ω_s relative to the central body.

The principal moments of inertia of the entire satellite in its undeformed state about the $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ axes are given by

$$A = A_c + A_r + 2A_p + 2m_p\ell^2$$

$$B = B_c + A_r + 2B_p$$

$$C = C_c + C_r + 2C_p + 2m_p\ell^2$$

The center of mass of the satellite is assumed to move in a circular orbit, with orbital angular speed Ω and radial distance R_0, about the center of a spherically symmetric planet. An equilibrium position is defined to be one in which the satellite is at rest with respect to the set of orbital axes $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ shown in Fig. 4. The orbital axes $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are defined as follows: 1) the origin is at the center of mass of the satellite, 2) $\boldsymbol{a}_1$ is in the radial direction, 3) $\boldsymbol{a}_2$ is tangent to the orbit in the direction of motion, and 4) $\boldsymbol{a}_3$ is normal to the orbit plane. The axes $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ of the satellite are oriented relative to the orbital axes by a 2-1-3 rotation sequence through θ_2, θ_1, and ψ, respectively (Fig. 5). It is shown in Ref. 14 that $\theta_2 = \theta_1 = \psi = 0$ is an equilibrium position of the satellite, and this is the equilibrium position used in this example.

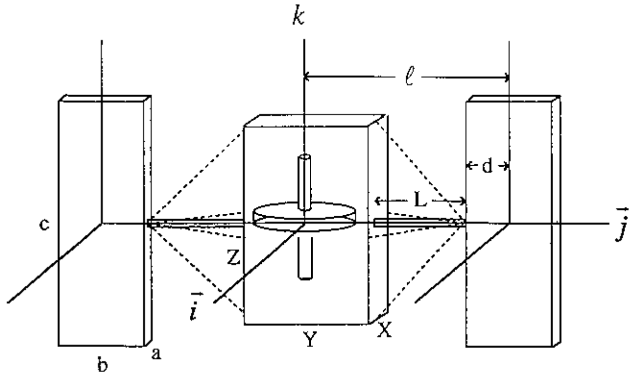

Fig. 3 Satellite model.

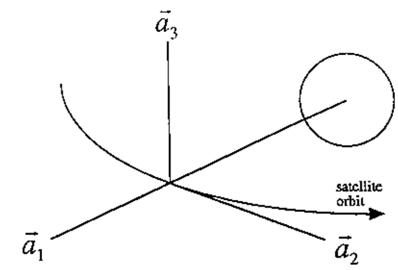

Fig. 4 Orbital axes.

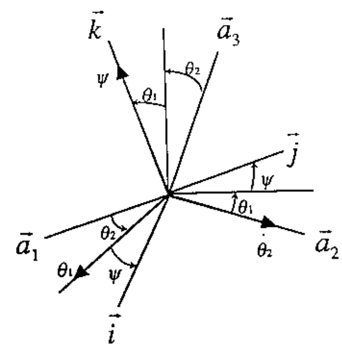

Fig. 5 Rotation sequence.

Stability Analysis

As derived in Ref. 13, D^2h_0 for the satellite is given by

$$D^2h_0 = \begin{bmatrix} h_{11} & 0 & 0 & 0 & 0 \\ 0 & h_{22} & 0 & h_{24} & -h_{24} \\ 0 & 0 & h_{33} & 0 & 0 \\ 0 & h_{24} & 0 & h_{44} & 0 \\ 0 & -h_{24} & 0 & 0 & h_{44} \end{bmatrix}$$

where

$$h_{11} = \left(C_c + C_r + 2C_p + 2m_p\ell^2 - B_c - A_r - 2B_p\right)\Omega^2 + C_r\omega_s\Omega$$

$$h_{22} = 4\left(C_c + C_r + 2C_p - A_c - A_r - 2A_p\right)\Omega^2 + C_r\omega_s\Omega$$

$$h_{24} = 4(A_p - C_p)\Omega^2$$

$$h_{33} = 3\left(B_c + 2B_p - A_c - 2A_p - 2m_p\ell^2\right)\Omega^2$$

$$h_{44} = \frac{JG}{L} - 4(A_p - C_p)\Omega^2$$

Sylvester's criterion is now applied to D^2h_0 and it is shown in Ref. 13 that sufficient conditions for stability are

1a) $h_{11} > 0$

1b) $h_{33} > 0$

1c) $h_{44} > 0$

1d) $h_{22}h_{44} - 2h_{24}^2 > 0$

and the expression for $\det D^2h_0$ is

$$\det D^2h_0 = h_{11}h_{33}h_{44}\left(h_{22}h_{44} - 2h_{24}^2\right)$$

The dominant stabilitycondition is $h_{22}h_{44} - 2h_{24}^2 > 0$.

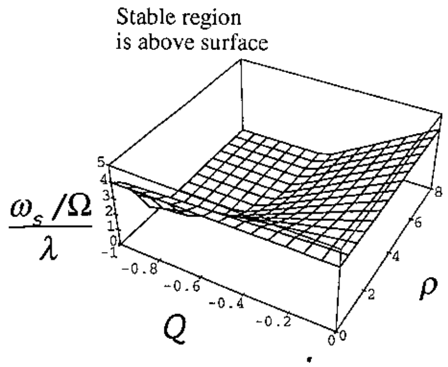

Fig. 6 Stability plot for example 3, panels down.

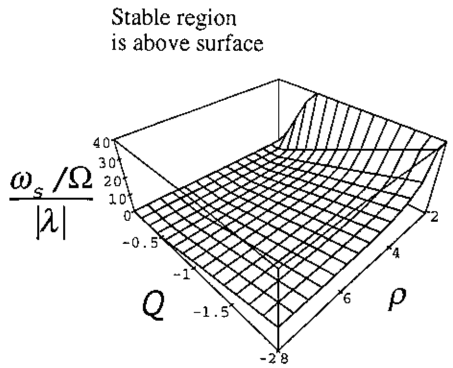

Fig. 7 Stability plot for example 3, panels up, $Q < 0$.

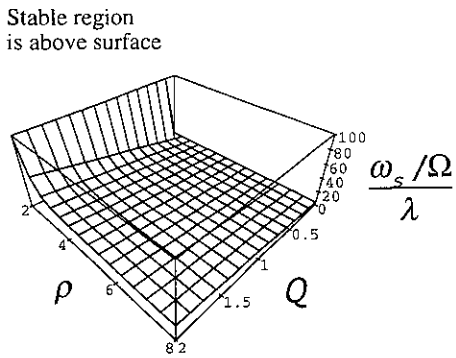

Fig. 8 Stability plot for example 3, panels up, $Q > 0$.

Several dimensionless parameters are needed in this example and they are defined as follows:

$\beta = (A_p - C_p)/B_p =$ dimensionless parameter

$\omega_n = (JG/LB_p)^{\frac{1}{2}} =$ natural frequency of each panel–shaft system

$\rho = \omega_n/\Omega =$ dimensionless parameter representing torsional stiffness

$Q = (A_p - C_p)/(A_c + A_r - C_c - C_r) =$ dimensionless parameter

$\lambda = (A_c + A_r - C_c - C_r)/C_r =$ dimensionless parameter

Note that when the panels are up (normal to the orbital plane), we have $\beta > 0$, and when the panels are down (lying in the orbital plane), we have $\beta < 0$. For thin panels, the magnitude of β is approximately one, so for the purpose of constructing stability plots we will set $\beta = 1$ when the panels are up and set $\beta = -1$ when the panels are down.

In terms of the dimensionless parameters defined earlier, it is shown in Ref. 12 that the preceding sufficient stability conditions 1a–1d can be written as

1a) $B > A$

1b) $\omega_s/\Omega > (B - C)/C_r$

1c) $\rho^2 > 4\beta$

1d) $\omega_s/\Omega > 4\lambda\{1 + 2Q[\rho^2/(\rho^2 - 4\beta)]\}$

Assuming that conditions 1a–1c are met, stability condition 1d is plotted in Figs. 6–8 for the three cases of panels down, panels up with $Q < 0$, and panels up with $Q > 0$. The plots show the minimum rotor spin rate required for given values of ρ and Q. Since the plots illustrate sufficient conditions derived via Lyapunov's direct method, the plotted surfaces are not guaranteed to separate stable and unstable regions. However, since $\det D^2h_0 = h_{11}h_{33}h_{44}(h_{22}h_{44} - 2h_{24}^2)$, it can be shown that the surfaces are part of set Z and so we can apply the methods of this paper to show that the surfaces are exact stability boundaries.

We check the sign of $\det D^2h_0$ at points below the surfaces in Figs. 6–8. We test the point ($Q = -0.2$, $\rho = 2$, $(\omega_s/\Omega)/\lambda = 1$) for panels down, and ($Q = -2$, $\rho = 4$, $(\omega_s/\Omega)/|\lambda| = 1$) and ($Q = 2$, $\rho = 4$, $(\omega_s/\Omega)/\lambda = 1$) for panels up. Evaluating the determinant of D^2h_0 at these points, we obtain values of $-597h_{11}h_{33}B_p^3\Omega^6$, $-1144h_{11}h_{33}B_p^3\Omega^6$, and $-1733h_{11}h_{33}B_p^3\Omega^6$, respectively. Since the signs of the determinants are negative at all these points, we have shown that the stability boundaries are exact.

The preceding analysis was carried out for a Hamiltonian satellite. However, as discussed earlier, the analysis remains valid if we add damping terms that have no linear dependence on position.

Conclusions

This paper develops simple tests to determine instability regions for general classes of systems in parameter space. Simplifications of the tests for even dimensional, Hamiltonian, second-order, and damped systems are derived. These simplifications greatly reduce the work in testing for instability regions. The tests can often be used in conjunction with Lyapunov's direct method to obtain exact stability boundaries in parameter space in cases where Lyapunov methods give only sufficient conditions for stability. In particular, if a region is shown to be stable by showing that the Hessian of the Hamiltonian is positive definite, then the typically occurring boundaries of this region can be shown to be exact using our methods without assumptions of complete or pervasive damping. The tests are simple enough to be a useful contribution to practical stability studies.

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