Approximating a loading-dependent cascading failure model with a branching process

Janghoon Kim, *Student Member, IEEE,* and Ian Dobson, *Fellow, IEEE*

***Abstract*—We quantify the closeness of the approximation between two high-level probabilistic models of cascading failure. In one model called CASCADE, failing components successively load the unfailed components, whereas the other model is based on a Galton-Watson branching process. Both models are generic, idealized models of cascading failure of a large, but finite number of components. For suitable parameters, the distributions of the total number of failures from the branching process and CASCADE models are close enough to make the branching process a useful approximation.**

***Index Terms*—Reliability theory, Probability**

NOMENCLATURE

$A(r,\theta,\lambda,n)$	Absolute difference $\lvert B(r,\theta,\lambda,n) - C(r,\theta/n,\lambda/n,n)\rvert$
$B(r,\theta,\lambda,n)$	Probability of r components failed in branching process
$C(r,d,p,n)$	Probability of r components failed in CASCADE
CASCADE	Loading-dependent cascading failure model
d	Initial disturbance in CASCADE
$G(r,\theta,\lambda)$	Generalized Poisson distribution
M_i	Number of components failed in generation i of branching process
n	Number of components
p	Load increment when a component fails in CASCADE
$Q(r,d,p,n)$	Quasibinomial distribution
$R(r,\theta,\lambda,n)$	Ratio $B(r,\theta,\lambda,n)/C(r,\theta/n,\lambda/n,n)$
$\overline{R}_1$	Upper bound on R in Lemma 1
$\overline{R}_2$	Upper bound on R in Lemma 2
$\underline{R}_3$	Lower bound on R in Lemma 3
$\overline{R}_{1\max}$	Upper bound on $\overline{R}_1$
$R_{\max}$	Upper bound on R in proofs
x	$((1-\lambda)r-\theta)/(n-r)$
θ	Mean initial failures in branching process
λ	Mean of offspring distribution in branching process

I. INTRODUCTION

Cascading failure is a sequence of s-dependent component failures that successively weaken a system. Here we consider aspects of cascading failure in large interconnected systems in which the large numbers of components makes it infeasible to enumerate the cascading failure sequences exhaustively. One example is electric power transmission systems which can contain tens of thousands of components that can interact in a variety of intricate ways [1], [2], [3]. Cascading failure of electric power systems causes large blackouts [1], [2]. High-level, idealized models of cascading failure, such as the CASCADE and branching process models considered in this paper, are useful in describing some essential features of observed and simulated cascading failure in these systems. We now give an introductory overview of these models.

The CASCADE model is an analytically tractable probabilistic model of cascading failure that captures the weakening of the system as the cascade proceeds [3]. There are a large but finite number n of identical components and each component has a level of loading or stress. The initial load on each component is an s-independent uniform random variable over a fixed range of loading. There is an initial disturbance to the system that adds additional loading to each component. Each component has a maximum loading threshold and fails if this threshold is exceeded. When any component fails, all the other components are additionally loaded so that initial failures can lead to a cascading sequence of failures as components successively overload and additionally load the other components. The cascade continues until there are no further failures or all the components are failed. The total number of failed components in the CASCADE model follows a saturating variant of the quasibinomial distribution. The main parameters are the size d of the initial disturbance and the amount p by which load of other components is incremented when a component fails, which controls the extent to which the cascade propagates.

The branching process model of cascading failure is a standard Galton-Watson branching process [4] with Poisson offspring distributions, except that there are a finite number n of components. The failures are produced in generations. In generation zero, there is an initial Poisson distribution of failures with mean θ that represents the initial disturbance to the system. Each failure in each generation produces further failures according to a Poisson offspring distribution with mean λ until no more failures are produced or all the components fail. The total number of failures follows a saturating variant of the generalized Poisson distribution. The main parameters are the mean size θ of the initial disturbance and the mean number of offspring failures λ which controls the extent to which the cascade propagates.

In pursuing high-level probabilistic models of cascading failure it is useful to know how different high-level models relate. It is known qualitatively that branching processes can approximate the CASCADE model in some parameter

Janghoon Kim and Ian Dobson are with the ECE Department, University of Wisconsin-Madison, Madison WI 53706 USA. (updated email is dobson@iastate.edu).

ranges (see section III-C and [5]). Also it is known that the two models are asymptotically the same when there are an infinite number of components [5], [6]. However, to make this approximation more useful in applications, it is necessary to consider the finite number of components and to quantify the approximation to be able to more precisely describe when the approximation is good. Particular reasons for wanting to apply this approximation are

1) The branching process model is simpler than the CASCADE model but the CASCADE model more directly summarizes a successive loading mechanism of cascading failure. It is advantageous to use the simpler branching process model when it is a good approximation.
2) There is a substantial and useful literature on applying branching processes to other cascading processes [4], [7], [8].
3) It is easier to determine when CASCADE and branching process models capture the main features of cascading failure data if the overlap and difference between CASCADE and branching processes is known precisely. For example, verifying one of the models in a particular cascading failure application automatically verifies the other model within the range in which they closely agree.

The purpose of this paper is to quantify how well a saturating Galton-Watson branching process model approximates the CASCADE model. In particular, explicit bounds are given for the closeness of the probability distributions of the total number of failures for the CASCADE and branching models. The paper completely reworks and refines some initial results in the MS thesis [9].

Sections II and III review the literature and specify and explain the CASCADE and branching process models. Section IV quantifies the closeness of the models and gives examples of the approximation. Detailed proofs are in Section V and Section VI concludes the paper.

II. Literature Review

The CASCADE model [3] was first introduced in [10], [11], and has close connections to fiber bundle models of material strength and waiting time models in queues [3]. The generalization to failures only causing further failures in a subset of other components is considered in [12]. Significant generalizations by Lefèvre and Gathy to inhomogeneous or random additional or initial loadings are in [13], [6]. The generalized CASCADE model can be related to the Reed-Frost model of epidemics [6]. The CASCADE model is related to simulated blackout data in [14], [15]. The total number of failed components in the CASCADE model follows a saturating variant of the quasibinomial distribution. The quasibinomial distribution was originally introduced for other purposes by Consul [16], [17], [18].

Branching processes are a standard model for cascades in many other subjects, including genealogy, cosmic rays, and epidemics [4]. Although branching processes are natural candidates for modeling cascading failure in risk analysis, this application of branching processes first appeared recently in [5], [19]. There is some initial evidence that branching process models can represent probability distributions of blackout size. Observed [20], [21] and simulated [22], [23], [24], [25], [21] blackout statistics show qualitative features such as probability distributions of blackout sizes with power law regions and criticality. These qualitative features can also be produced by saturating branching processes [5]. Moreover, branching processes have approximately reproduced the distribution of blackout sizes obtained from data or initial simulations [26], [27], [28], [29]. Branching processes well approximate epidemic models similar to cascading models as the number of susceptible individuals grows large [30], [31], [32], [6].

Chen and McCalley describe an accelerated propagation model for the number of transmission line failures in [33]. For parameters based on combined data for North American transmission line failures from [34], the accelerated propagation model applies to up to 7 failures. They examine the fit of the accelerated propagation model, a generalized Poisson distribution, and a negative binomial distribution to the North American transmission line failure data. Both the accelerated propagation model and the generalized Poisson distribution are consistent with the data.

There is an extensive literature on cascading in graphs [35], [36] that is motivated in part by propagation of failures in the internet. The dynamics of cascading is related to statistical properties of the graph topology. Work on cascading phase transitions and network vulnerability that accounts for network loading includes Watts [37], Motter [38], Crucitti [39] and Lesieutre [40]. Roy [41] considers Markov models for reliability on abstract influence graphs.

There are some general approaches to cascading failure risk for systems with a modest number of components. Sun [42] and Lindley [43] represent cascading failure by increasing the failure rate of remaining components when a component fails. Sun [42] applies accelerating failure to gradual degradation of a mechanical system.

Simulations of cascading failure blackouts in electric power systems are reviewed in [21], [44]. These simulations approximate the physics of some selection of the actual cascading mechanisms and compute some possible cascading sequences for a set of initial conditions. The network and the patterns of power flow and power injection change according to circuit laws and operational procedures as power system components fail. The changing network structure as cascading failure proceeds is also represented by Greig [45] in more general flow networks. Cascading blackouts pose substantial challenges to risk analysis because of the large size of the networks and the complexity and variety of the cascading mechanisms and interactions. A detailed direct analysis is intractable and even simulation approaches are greatly simplified. High-level probabilistic models are a useful complement to the more detailed models and simulations.

III. CASCADE and branching process models

This section states the CASCADE and branching process cascading failure models and illustrates their qualitative agreement.

A. CASCADE

The normalized form of the CASCADE model has n identical components with random and s-independent initial loads that are uniformly distributed in $[0,1]$. Components fail when their load exceeds 1. When a component fails, a fixed amount of load $p \geq 0$ is transferred to each of the components. To start the cascade, there is an initial disturbance that loads each component by an additional amount $d \geq 0$. Then components whose loads exceed 1 fail, and the failure of any of these components will distribute an additional load p to all the components that can cause further failures in a cascade. The model can be defined more precisely in algorithmic form:

Normalized CASCADE algorithm [3]

(0) All n components are initially unfailed and have initial loads that are n s-independent random variables uniformly distributed in $[0,1]$.
(1) Add the initial disturbance d to the load of each component. Initialize the generation number i to zero.
(2) Test each unfailed component for failure: For $j = 1,...,n$, if component j is unfailed and its load > 1 then component j fails. Suppose that m_i components fail in this step. If $m_i = 0$, stop.
(3) Increment the component loads according to the number of failures m_i: Add $m_i p$ to the load of each component.
(4) Increment generation number i and go to step 2.

In this paper we are interested in systems with a large number of components, but for purposes of illustration we show in Table I a simple example of one realization of the CASCADE model with 5 components producing a cascade. Fig. 1 shows the succession of load increases in this cascade labelled with their generation number.

TABLE I
COMPONENT LOADS INCREASING IN A SMALL EXAMPLE OF CASCADE

$n = 5$ components
initial disturbance $d = 0.3$
load increment $p = 0.1$
generation number i

	component number					
i	1	2	3	4	5	
0	0.8	0.6	0.75	0.45	0.1	initial random loads
1	1.1	0.9	1.05	0.75	0.4	initial disturbance d added
2	1.3	1.1	1.25	0.95	0.6	1 and 3 fail; $2p$ added
3	1.4	1.2	1.35	1.05	0.7	2 fails; p added
4	1.5	1.3	1.45	1.15	0.8	4 fails; p added

This cascade ends with components 1,2,3,4 failed.

The total number of failures in the normalized CASCADE model has the probability distribution [3]

$$C(r,d,p,n) = \begin{cases} \binom{n}{r} d(rp+d)^{r-1}(\max\{1-rp-d,0\})^{n-r}, & 0 \leq r < n, \\ 1 - \sum_{s=0}^{n-1} C(s,d,p,n), & r = n. \end{cases} \tag{1}$$

If $np + d > 1$, note that $C(r,d,p,n) = 0$ for $\frac{1-d}{p} \leq r < n$.

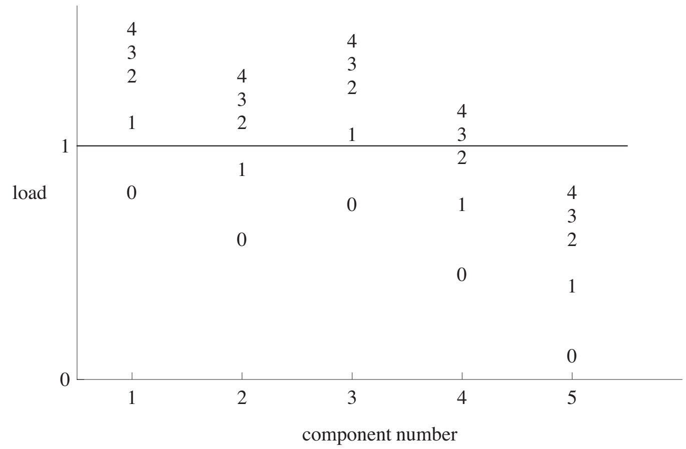

Fig. 1. Component loads increasing in the small example cascade with 5 components described in Table I. The random initial loads in generation 0 are shown by zeros, the generation 1 loads are shown by ones, and the generation i loads are shown by the numeral i. The loads increase due to the initial disturbance $d = 0.3$ or due to the load added to each component $p = 0.1$ for each component that fails.

For $p = 0$, there is no cascade propagation after the initial disturbance and (1) reduces to the binomial distribution. Fig. 2 shows examples of the probability distribution (1) as p increases in the case of $n = 5000$ components and small initial disturbance $d = 0.0002$. The distribution for $p = 0.0001$ and $np = 0.5$ has an exponential tail slightly heavier than binomial. Hence there is an extremely small probability of cascades in which a large fraction of the components fail. The tail becomes heavier as p increases and the distribution for $p = 0.0002$ and $np = 1$ has an approximate power law region over a range of r. This implies a non negligible probability of cascades that extend to the system size, and, in this case, the probability of all 5000 components failing is 0.00054. The distribution for $p = 0.0003$ and $np = 1.5$ has an approximately exponential tail for small r, zero probability of intermediate r, and a probability of 0.44 of all 5000 components failing. (If an intermediate number of components fail, then the cascade always proceeds to all 5000 components failing.)

For $np+d < 1$ and $r < n$, (1) reduces to the quasibinomial distribution [16], [17], [18]:

$$Q(r,d,p,n) = \binom{n}{r} d(rp+d)^{r-1}(1-rp-d)^{n-r}, \tag{2}$$
$$0 \leq r \leq n \text{ and } np+d < 1.$$

The CASCADE model and (1) extend the quasibinomial distribution to stressed systems with $np + d \geq 1$ that have a high probability of all components failing, such as the case $np = 1.5$ shown in Fig. 2.

Fig. 3 shows the probability distribution of the total number of failures for $p = 0.0001$ and $np = 0.5$, but with a larger initial disturbance $d = 0.0002$ that gives a mean initial disturbance of $nd = 10$ components failed.

In the CASCADE model, each failure adds load to all the other components, but a variant of CASCADE in which load is added to a randomized subset of other components has been studied [12]. When applying the CASCADE model, it is common to use an unnormalized version of CASCADE, which allows free choices of the load at which components fail and the upper and lower bounds on the initial distribution of loading [3]. Since the unnormalized and normalized versions only differ by a rescaling of parameters and have identical probability distributions, it is convenient in this paper to only consider the normalized version of CASCADE.

B. Branching process

The branching process model for cascading failure [5], [19], [27], [26] is a standard Galton-Watson branching process, except that the process saturates when all n components fail. The failures are produced in generations. The initial number of failures M_0 in generation zero is given by a Poisson distribution of mean $\theta \geq 0$, except that it saturates at n components:

$$P[M_0 = m] = \begin{cases} \dfrac{\theta^m}{m!}\, e^{-\theta}, & 0 \leq m < n, \\ 1 - \displaystyle\sum_{s=0}^{n-1} \frac{\theta^s}{s!}\, e^{-\theta}, & m = n. \end{cases} \tag{3}$$

In subsequent generations, if there are M_i failures in generation i and $0 < M_i < n$, then the kth failure in generation i produces $M_{i+1}^{(k)}$ failures in generation $i+1$ according to a Poisson distribution of mean $\lambda \geq 0$:

$$P[M_{i+1}^{(k)} = m] = \frac{\lambda^m}{m!}\, e^{-\lambda}, \quad m \geq 0, \tag{4}$$

where $M_{i+1}^{(1)}, M_{i+1}^{(2)}, \cdots, M_{i+1}^{(M_i)}$ are s-independent. Then the number of failures in generation $i+1$ is

$$\begin{aligned} M_{i+1} = \min\big\{&M_{i+1}^{(1)} + M_{i+1}^{(2)} + \cdots + M_{i+1}^{(M_i)}, \\ &n - (M_0 + M_1 + ... + M_i)\big\}. \end{aligned} \tag{5}$$

In (5), $n - (M_0 + M_1 + ... + M_i)$ is the remaining number of unfailed components after generation i. In any generation, if the number of failures is $M_i = 0$ or $M_i = n$, then the cascade stops.

The use of the Poisson distribution as a suitable approximation of the offspring distribution can be derived from an assumption that the failures propagate so that each failure has a small uniform probability of independently causing failure in a large number of other components [18]. In modeling cascading failure with the branching process, we do not imply that each failure in some sense “causes” its offspring failures in the next generation; the branching process simply produces random numbers of failures in each generation that can statistically match the outcomes of the cascading process. The modeling of saturation is further considered in [27], [5], [15].

The total number of failures in the Galton-Watson process with an unlimited number of components has a generalized Poisson distribution for $0 \leq \lambda < 1$ [17], [18]. It follows that the distribution of the total number of failures in our branching process model with a finite number of components n is the following saturating variant of the generalized Poisson distribution:

$$B(r, \theta, \lambda, n) = \begin{cases} \theta(r\lambda + \theta)^{r-1} \dfrac{e^{-r\lambda-\theta}}{r!}, & r < n, \\ 1 - \displaystyle\sum_{s=0}^{n-1} B(s, \theta, \lambda, n), & r = n. \end{cases} \tag{6}$$

For $0 \leq \lambda < 1$, and for an infinite number of components n, (6) becomes the generalized Poisson distribution [17], [18]:

$$G(r, \theta, \lambda) = \theta(r\lambda + \theta)^{r-1} \frac{e^{-r\lambda-\theta}}{r!}, \tag{7}$$
$$0 \leq r < \infty \text{ and } 0 \leq \lambda < 1.$$

The saturating branching process and (6) model a finite number of components and enable the study of stressed systems with $\lambda \geq 1$ that have a high probability of all components failing, such as the case $\lambda = 1.5$ shown in Fig. 4.

C. Qualitative agreement and asymptotic agreement between models

The parameters of the CASCADE and branching process models correspond according to

$$\theta = nd \tag{8}$$
$$\lambda = np \tag{9}$$

This can be understood as follows: At generation zero of the CASCADE model, the components that will fail when the initial disturbance d is added are exactly those that have loading in $(1-d, 1]$. Therefore, the probability of any particular component failing when the initial disturbance is added is d and the mean number of initial failures is nd. Since the mean number of initial failures in the branching process is θ, we obtain (8). To similarly obtain (9), observe that each failure in CASCADE causes a load increment of p. Then an approximate result is that the mean number of failures of other components due to the load increment of p is np. This approximation is valid for large n and small p and is examined in detail in [5]. Since the mean number failures in the next generation due to one failure in the branching process is λ, we obtain (9).

Comparing Figs. 2 and 4 illustrates the qualitative agreement between the CASCADE and branching process models. The model parameters chosen in Figs. 2 and 4 correspond according to (8) and (9). Also note the difference between the models for $np = \lambda = 1.0$ and more than 3000 components failed. Plotting the branching process probability distribution corresponding to the CASCADE distribution in Fig. 3 yields a figure indistinguishable from Fig. 3.

Now we consider the asymptotic agreement between models. Suppose that the number of components $n \to \infty$, with $d = \theta/n \to 0$ and $p = \lambda/n \to 0$ so that θ and λ given by (8) and (9) remain constant. Then the quasibinomial distribution (2) tends to the generalized Poisson distribution (7). Indeed, Consul [17] proved that, as $n \to \infty$,

$$\begin{aligned} Q(r, \frac{\theta}{n}, \frac{\lambda}{n}, n) = G(r, \theta, \lambda) \left(1 + \frac{r - (r(1-\lambda) - \theta)^2}{2n}\right. \\ \left. + O(n^{-2})\right). \end{aligned}$$

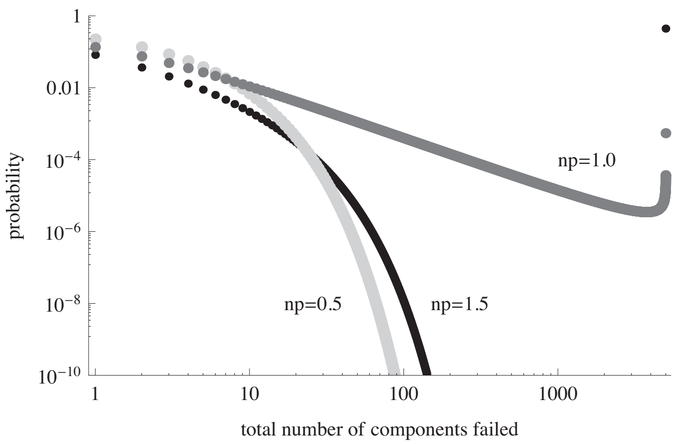

Fig. 2. Probability distribution of total number of failures from CASCADE model with $n = 5000$ components, initial disturbance d with $nd = 1$ and each failure causing load increment p with $np = 0.5$ (light gray dots), $np = 1.0$ (dark gray dots), and $np = 1.5$ (black dots). The probability of 5000 failures is negligible for $np = 0.5$, 0.00054 for $np = 1$, and 0.44 for $np = 1.5$. The probability of zero failures is 0.3678 in all cases.

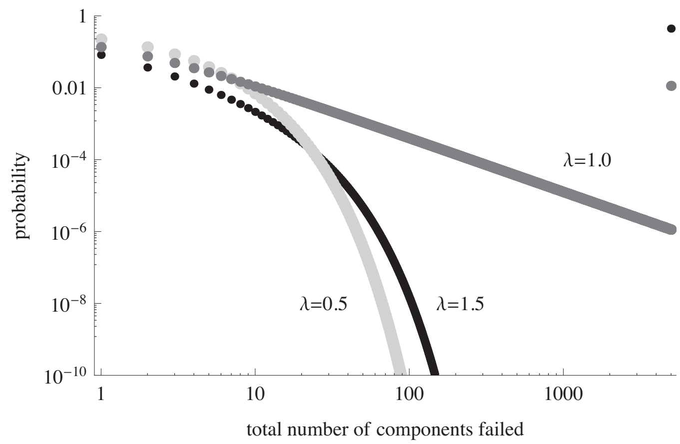

Fig. 4. Probability distribution of total number of failures from branching process model with $n = 5000$ components, initial disturbance $\theta = 1$ and offspring mean $\lambda = 0.5$ (light gray dots), $\lambda = 1.0$ (dark gray dots), and $\lambda = 1.5$ (black dots). The probability of 5000 failures is negligible for $\lambda = 0.5$, 0.011 for $\lambda = 1$, and 0.44 for $\lambda = 1.5$. The probability of zero failures is 0.3679 in all cases.

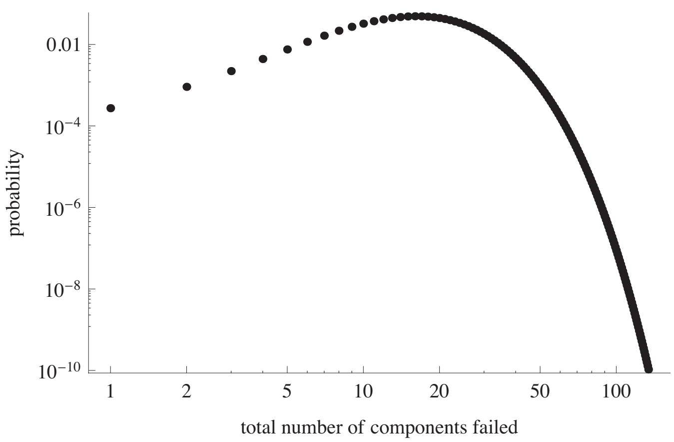

Fig. 3. Probability distribution of total number of failures from CASCADE model with $n = 5000$ components, initial disturbance d with $nd = 10$ and each failure causing load increment p with $np = 0.5$. The probability of 5000 failures is negligible. The probability of zero failures is 0.000045.

It follows that the CASCADE distribution (1) tends to the branching process distribution (6) as $n \to \infty$. This is a basic result relating CASCADE with its branching process approximation. However, it does not give quantitative bounds for our intended applications with a finite number of components n.

IV. Quantifying closeness of models

To quantify the difference between the CASCADE and branching process models when their parameters correspond according to (8) and (9), we define the ratio and absolute difference:

$$R(r,\theta,\lambda,n) = \frac{B(r,\theta,\lambda,n)}{C(r,\theta/n,\lambda/n,n)} = \frac{(n-r)!n^n e^{-r\lambda-\theta}}{n!(n-r\lambda-\theta)^{n-r}}, \quad (10)$$

$$0 \le r < \min\left\{\frac{n-\theta}{\lambda}, n\right\}. \quad (11)$$

(The condition $r < \frac{n-\theta}{\lambda}$ ensures that $C(r,\theta/n,\lambda/n,n) \neq 0$. If $0 \le \lambda \le 1 - \frac{\theta}{n}$, then $n \le \frac{n-\theta}{\lambda}$ and (10) is valid for $0 \le r < n$.)

$$A(r,\theta,\lambda,n) = \left|B(r,\theta,\lambda,n) - C(r,\frac{\theta}{n},\frac{\lambda}{n},n)\right| \quad (12)$$

Now we state lemmas bounding these differences. All proofs are postponed to section V.

For the ratio R in (10), we have

Lemma 1: Suppose that $0 \le \lambda < 1$ and $\theta > 0$. Then

$$R(r,\theta,\lambda,n) < \overline{R}_1 = \exp\left(\frac{(r(1-\lambda)-\theta)^2}{2(n-r)}\right)$$
$$\text{for} \quad \frac{\theta}{1-\lambda} \le r < \min\left\{\frac{n-\theta}{\lambda}, n\right\}.$$

Lemma 2: Suppose that $\lambda > 1/3$ and $0 \le \theta \le 2n/3$. Then

$$R(r,\theta,\lambda,n) < \overline{R}_2 = \exp\left(\frac{(r(1-\lambda)-\theta)^2}{n-r}\right)$$
$$\text{for} \quad 0 \le r < \min\left\{\frac{\frac{2}{3}n-\theta}{\lambda-\frac{1}{3}}, n\right\}.$$

Lemma 3: Suppose that $\lambda > 0$ and $0 < \theta < n$. Then

$$R(r,\theta,\lambda,n) > \underline{R}_3 = \sqrt{1-\frac{r}{n}}$$
$$\text{for} \quad 0 \le r < \min\left\{\frac{n-\theta}{\lambda}, n\right\}.$$

The bounds on the ratio R in Lemmas 1, 2 and 3 yield bounds on the absolute difference A in (12):

Lemma 4:

(i) Suppose that $0 \le \lambda < 1$ and $0 < \theta < n$. Then

$$A(r,\theta,\lambda,n) < B(r,\theta,\lambda,n) \max\{\underline{R}_3^{-1} - 1, 1 - \overline{R}_1^{-1}\}$$
$$\text{for} \quad \frac{\theta}{1-\lambda} \le r < \min\left\{\frac{n-\theta}{\lambda}, n\right\}.$$

(ii) Suppose that $\lambda > 1/3$ and $0 \le \theta \le 2n/3$. Then

$$A(r,\theta,\lambda,n) < B(r,\theta,\lambda,n) \max\{\underline{R}_3^{-1} - 1, 1 - \overline{R}_2^{-1}\}$$
$$\text{for} \quad 0 \le r < \min\left\{\frac{\frac{2}{3}n-\theta}{\lambda - \frac{1}{3}}, n\right\}.$$

It is often the case that $B > C$ for large r:

Lemma 5: Suppose that $n > 100$, $0 \le \lambda < 2$, and $\theta > 0$. Then

$$B(r,\theta,\lambda,n) > C(r,\theta/n,\lambda/n,n)$$
$$\text{for} \quad \frac{\sqrt{n}+\theta}{|1-\lambda|} < r < \min\left\{n, \frac{n+\theta}{2-\lambda}\right\}. \qquad (13)$$

Moreover, if either $0 \le \lambda < 0.92 - \frac{2\theta}{n}$ or $1 - \frac{\theta}{n} \le \lambda < 2$, then

$$B(r,\theta,\lambda,n) > C(r,\theta/n,\lambda/n,n)$$
$$\text{for} \quad \frac{\sqrt{n}+\theta}{|1-\lambda|} < r < n. \qquad (14)$$

We also note the monotonicity properties of B:

Lemma 6:
(i) If $0 \le \lambda < 1$ and $\frac{\theta+1}{1-\lambda} \le r < n$, then $B(r,\theta,\lambda,n)$ is increasing in λ.
(ii) If $\lambda \ge 1$ and $0 \le r < n$, then $B(r,\theta,\lambda,n)$ is decreasing in λ.
(iii) If $0 \le \lambda < 1$ and $\frac{\theta-1}{1-\lambda} \le r < n$, then $B(r,\theta,\lambda,n)$ is decreasing in r.

We now illustrate the use of the Lemmas.

A. Example 1

In our motivating application of cascading failure blackouts in power transmission networks, estimates for the costs vary widely. For example, estimates for direct costs of the August 2003 blackout of Northeastern America vary from about 4 to 12 billion dollars. And indirect costs, such as when there is rioting or damage to other infrastructures, can readily double or triple the costs, but are uncertain and hard to quantify. Suppose that risk is computed as probability of blackout times cost. Then there is little use for estimates of blackout probability that are significantly more accurate than the costs. For the sake of illustration, we measure the cascading blackout size by the number of failures and require blackout probabilities to be accurate within a factor of 2. That is, we require our branching process approximation to have ratio R in (10) satisfy $\frac{1}{2} < R < 2$.

Assume that $0 < \lambda < 1$, $r \le \frac{n}{2}$, and $\theta \le \frac{n}{2}$ (note that $\theta \le \frac{n}{2}$ and $0 < \lambda < 1$ imply $\frac{n-\theta}{\lambda} > \frac{n}{2}$). Then the upper bound

$$\overline{R}_1 = \exp\left(\frac{(r(1-\lambda)-\theta)^2}{2(n-r)}\right)$$

from Lemma 1 is bounded above by $\overline{R}_{1\max}$, where

$$\overline{R}_{1\max} = \exp\left(\frac{(r(1-\lambda)-\theta)^2}{n}\right).$$

And

$$r < \frac{0.83\sqrt{n}+\theta}{1-\lambda} \Rightarrow \overline{R}_{1\max} < 2.$$

Moreover, Lemma 3 implies that $R > \frac{1}{2}$ when $r \le \frac{3n}{4}$. We conclude that $0 < \lambda < 1$ and $\theta \le \frac{n}{2}$ and

$$\frac{\theta}{1-\lambda} < r < \min\left\{\frac{0.83\sqrt{n}+\theta}{1-\lambda}, \frac{n}{2}\right\} \Rightarrow \frac{1}{2} < R < 2.$$

The range over which the approximation is valid increases with n.

B. Example 2

Practical industry models for power transmission networks typically range from hundreds to tens of thousands of nodes. We choose $n = 1000$ nodes, a small initial disturbance $\theta = 1$ and $\lambda = 0.5$. Then the Lemma 1 upper bound $\overline{R}_1 < 2$ for $2 \le r \le 73$ and the Lemma 3 lower bound $\underline{R}_3 > \frac{1}{2}$ for $r \le 750$. That is, Lemmas 1 and 3 imply that $\frac{1}{2} < R < 2$ for $2 \le r \le 73$. The Lemma 2 upper bound $\overline{R}_2 < 2$ for $0 \le r \le 53$. Combining the results from Lemmas 1, 2 and 3, we obtain $\frac{1}{2} < R < 2$ for $0 \le r \le 73$. This bound is fairly tight: direct calculation shows that in this case, the maximum range of r over which $\frac{1}{2} < R < 2$ is $0 \le r \le 76$.

The probability of 73 total failures in the branching model is $B(73,1,0.5,1000) = 2.48 \times 10^{-9}$. Lemma 5 shows that $B > C$ for $r \ge 66$ and Lemma 6 shows that $B(r,1,0.5,1000)$ is decreasing in r, except for $r = 1000$. Therefore the Lemmas yield $\frac{1}{2} < R < 2$ for $0 \le r \le 73$ and $C < B < 2.48 \times 10^{-9}$ for $73 \le r < 1000$.

To show the effect of increasing n, redoing Example 2 with $n = 10000$ nodes yields $\frac{1}{2} < R < 2$ for $0 \le r \le 234$ and $C < B < 1.40 \times 10^{-23}$ for $234 \le r < 1000$.

C. Example 3

We choose $n = 1000$, $\theta = 1$ and $\lambda = 0.98$. The Lemma 1 upper bound $\overline{R}_1 < 2$ for $50 \le r \le 826$, the Lemma 2 upper bound $\overline{R}_2 < 2$ for $r \le 731$, and the Lemma 3 lower bound $\underline{R}_3 > \frac{1}{2}$ for $r \le 750$. Therefore $\frac{1}{2} < R < 2$ for $0 \le r \le 750$. For comparison with these bounds, R is plotted as a function of r in Figure 5. Note the rapid increase in R for $r > 800$. The probability of 750 failures in the branching model is $B(750,1,0.98,1000) = 8.15 \times 10^{-6}$ and Lemma 6 shows that $B(r,1,0.98,1000)$ is decreasing in r for $0 \le r < 1000$.

Redoing Example 3 with $n = 10000$ nodes yields $\frac{1}{2} < R < 2$ for $0 \le r \le 4439$.

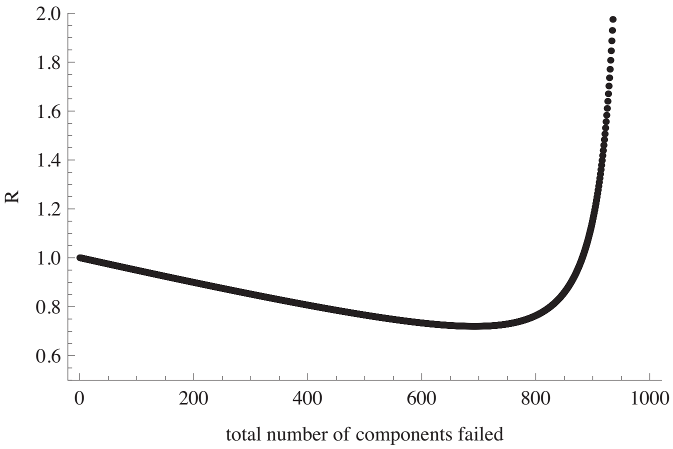

Fig. 5. Ratio $R(r,\theta,\lambda,n)$ of branching and CASCADE probabilities for $n = 1000$, $\theta = 1$, and $\lambda = 0.98$.

V. Proofs

A. Proof of Lemma 1

The Stirling approximation

$$\sqrt{2n\pi}n^n e^{-n+\frac{1}{12n+1}} < n! < \sqrt{2n\pi}n^n e^{-n+\frac{1}{12n}} \tag{15}$$

[46] applied to (10) yields

$$\begin{aligned} &R(r,\theta,\lambda,n) < R_{\max}, \\ &\quad \text{where } R_{\max} = \sqrt{1-r/n}\,(n-r)^{n-r}e^{r+\frac{1}{12(n-r)}-\frac{1}{12n+1}} \\ &\qquad\qquad\qquad \times (n-\lambda r-\theta)^{r-n}e^{-\lambda r-\theta}. \end{aligned}$$

We have $n > r \geq \frac{\theta}{1-\lambda} > 0$. Therefore $n - r \geq 1$ and $r \geq 1$ and

$$\tfrac{1}{2}\ln\left[1-\frac{r}{n}\right] + \frac{\frac{1}{12}(r+\frac{1}{12})}{(n-r)(n+\frac{1}{12})} < -\frac{r}{2n} + \frac{r+\frac{1}{12}}{12n} < 0.$$

Hence

$$\ln R_{\max} < (1-\lambda)r-\theta-(n-r)\ln\left[1+\frac{(1-\lambda)r-\theta}{n-r}\right] \tag{16}$$

and, writing

$$x = \frac{(1-\lambda)r-\theta}{n-r}, \tag{17}$$

$$\ln R_{\max} < (n-r)\left(x-\ln[1+x]\right). \tag{18}$$

Then $\frac{\theta}{1-\lambda} \leq r < n$ implies $x \geq 0$. Since

$$x - \ln[1+x] \leq \frac{x^2}{2} \quad \text{for} \quad x \geq 0,$$

$$\ln R < \ln R_{\max} \leq \frac{((1-\lambda)r-\theta)^2}{2(n-r)}.$$

B. Proof of Lemma 2

First we check that r is in the range (11) in which R is defined. If $\lambda \leq 1-\frac{\theta}{n}$, then $r < n \leq \frac{n-\theta}{\lambda}$. If $\lambda > 1-\frac{\theta}{n}$, then $r < \frac{\frac{2}{3}n-\theta}{\lambda-\frac{1}{3}} < \frac{n-\theta}{\lambda}$.

Now consider the case $r = 0$. $\theta \leq 2n/3$ implies that $-\theta/n \geq -2/3$, and since

$$x - \ln[1+x] < x^2 \quad \text{for} \quad x > -\frac{2}{3}, \tag{19}$$

$$\ln R(0,\theta,\lambda,n) = -\theta - n\ln\left[1-\frac{\theta}{n}\right] < \frac{\theta^2}{n}.$$

The proof of Lemma 2 for $r \geq 1$ is similar to the proof of Lemma 1 up to and including (18). Then $r < \frac{\frac{2}{3}n-\theta}{\lambda-\frac{1}{3}}$ implies $x > -\frac{2}{3}$. And (18) and (19) yield

$$\ln R < \ln R_{\max} < (n-r)x^2 \leq \frac{((1-\lambda)r-\theta)^2}{n-r}.$$

C. Proof of Lemma 3

In the case $r = 0$,

$$\ln R(0,\theta,\lambda,n) = -\theta - n\ln\left[1-\frac{\theta}{n}\right] > 0.$$

Now consider the case $r \geq 1$. Using (15),

$$\begin{aligned} R(r,\theta,\lambda,n) &> \sqrt{\frac{n-r}{n}}\left(1+\frac{r(1-\lambda)-\theta}{n-r}\right)^{r-n} \\ &\quad \times \exp\left[r(1-\lambda)-\theta-\frac{1}{12n}+\frac{1}{12(n-r)+1}\right]. \end{aligned}$$

$r \geq 1$ implies $\frac{1}{12n} < \frac{1}{12(n-r)+1}$ so that

$$R > \sqrt{\frac{n-r}{n}}\left(\frac{e^x}{1+x}\right)^{n-r}, \tag{20}$$

where x is given in (17).

Now $r < \frac{n-\theta}{\lambda}$ implies $x > -1$. Since

$$x - \ln[1+x] \geq 0 \quad \text{for} \quad x > -1,$$

$$R > \sqrt{1-\frac{r}{n}}.$$

D. Proof of Lemma 4

$0 < B(r,\theta,\lambda,n) < 1$ and $0 \leq C(r,\theta/n,\lambda/n,n) < 1$. Therefore

$$\begin{aligned} A = |B-C| &< B\left|1-\frac{C}{B}\right| \\ &\leq B\max\left\{|1-\min\{\frac{C}{B}\}|, |\max\{\frac{C}{B}\}-1|\right\}. \end{aligned}$$

Applying lemmas 1 and 3 or lemmas 1 and 2 yields the result. (For (ii), note that if $\lambda \leq 1-\frac{\theta}{n}$, then $n \leq \frac{n-\theta}{\lambda}$, whereas if $\lambda > 1-\frac{\theta}{n}$, then $\frac{\frac{2}{3}n-\theta}{\lambda-\frac{1}{3}} < \frac{n-\theta}{\lambda}$.)

E. Proof of Lemma 5

Noting that $r > \frac{\sqrt{n}+\theta}{|1-\lambda|}$ implies $r \geq 1$, the proof of Lemma 5 is the same as the proof of Lemma 3 for $r \geq 1$ up to and including (20). If $n - \lambda r - \theta \leq 0$, then $B > 0 = C$ and the lemma is proved. Therefore in the following we assume $n - \lambda r - \theta > 0$, or, equivalently, $x > -1$.

(i) Consider the case $r \leq \frac{n+\theta}{2-\lambda}$. In this case, $x \leq 1$. Since $x - \ln[1+x] \geq \frac{x^2}{2e}$ for $-1 < x < 1$, (20) becomes

$$R > \sqrt{\frac{n-r}{n}} \exp\left[\frac{(r(\lambda-1)+\theta)^2}{2e(n-r)}\right].$$

$r > \frac{\sqrt{n}+\theta}{|1-\lambda|}$ yields $(r(\lambda-1)+\theta)^2 > n$. Then

$$R > \sqrt{\frac{n-r}{n}} \exp\left[\frac{n}{2e(n-r)}\right] \geq 1,$$

where the second inequality is verified by squaring both sides, writing $\frac{n}{e(n-r)} = t$, and using $\ln t \leq 1 - t$ for $t > 0$.

(ii) Consider the case $1 - \frac{\theta}{n} \leq \lambda < 2$. Then $n \leq \frac{n+\theta}{2-\lambda}$ and (13) becomes (14).

(iii) Consider the case $r > \frac{n+\theta}{2-\lambda}$ and $\lambda < 0.92 - \frac{2\theta}{n}$. In this case, $x > 1$. Since $\ln[1+x] - 0.7x < 0$ for $x > 1$, (20) becomes

$$R > \sqrt{\frac{n-r}{n}}\, e^{0.3(r(1-\lambda)-\theta)}.$$

Define $f(r) = 0.5 \ln \frac{n-r}{n} + 0.3(r(1-\lambda)-\theta)$. It is straightforward to check that $f(r)$ is concave. Moreover, since $f(\frac{n}{2}) > 0$ and $f(n-1) > 0$ for $\lambda < 0.92 - \frac{2\theta}{n}$ and $n \geq 100$, $f(r) > 0$ for $\frac{n}{2} \leq r \leq n-1$. Therefore, since $r > \frac{n+\theta}{2-\lambda} \geq \frac{n}{2}$, $R > 1$.

F. Proof of Lemma 6

(i) and (ii)

$$\frac{\partial B}{\partial \lambda} = \frac{\theta}{r!} r(r\lambda+\theta)^{r-2} e^{-r\lambda-\theta}(r(1-\lambda)-(1+\theta))$$

(iii)

$$\begin{aligned}\frac{B(r+1,\theta,\lambda,n)}{B(r,\theta,\lambda,n)} &= \frac{((r+1)\lambda+\theta)^r}{(r+1)(r\lambda+\theta)^{r-1}} e^{-\lambda} \\ &= (\lambda + \frac{\theta}{r+1})\left(\left(1+\frac{\lambda}{r\lambda+\theta}\right)^{r\lambda+\theta}\right)^{\frac{r-1}{r\lambda+\theta}} e^{-\lambda} \\ &< (\lambda+\frac{\theta}{r+1}) e^{1-\lambda-\frac{\theta+\lambda}{r\lambda+\theta}} \\ &\leq (\lambda+\frac{\theta}{r+1}) e^{1-\left(\lambda+\frac{\theta}{r+1}\right)} e^{-\frac{\lambda}{r+1}} < 1,\end{aligned} \tag{21}$$

where (21) follows since $(1-\lambda)r - \theta \geq -1$ implies $r+1 \geq r\lambda+\theta$ and since $te^{1-t} \leq 1$ for $t > 0$.

VI. Conclusion

High-level probabilistic models of cascading failure such as the CASCADE model are emerging as one of the useful approaches in the study of large blackouts and may find future testing and potential application in cascading failure of other large, interconnected systems. In this paper we approximate CASCADE with a Galton-Watson branching process and give quantitative bounds on the closeness of the approximation for the probability distribution of the total number of failures. Since the branching process is a simple and well understood probabilistic model, it is advantageous to use it when it is a good approximation. The analysis accounts for the large but finite number of components needed in the intended engineering applications.

Acknowledgments

This paper is an account of work sponsored in part by the Power Systems Engineering Research Center (PSERC). We gratefully acknowledge support in part from NSF grants ECCS-0606003 and SES-0623985 and DOE grant DE-SC0002283. J. Kim gratefully acknowledges support in part from the Korean electric power corporation.

References

[1] U.S.-Canada Power System Outage Task Force, *Final report on the August 14th blackout in the United States and Canada*. United States Department of Energy and National Resources Canada, April 2004.

[2] IEEE PES PSDP Task Force on Blackout experience, mitigation, and role of new technologies, blackout experiences and lessons, Best practices for system dynamic performance, and the role of new technologies, IEEE Special Publication 07TP190, July 2007.

[3] I. Dobson, B.A. Carreras, D.E. Newman, A loading-dependent model of probabilistic cascading failure, Probability in the Engineering and Informational Sciences, vol. 19, no. 1, January 2005.

[4] T.E. Harris, *Theory of branching processes*, Dover NY 1989.

[5] I. Dobson, B.A. Carreras, D.E. Newman, A branching process approximation to cascading load-dependent system failure, 37th Hawaii International Conference on System Sciences, Hawaii, January 2004.

[6] M. Gathy, C. Lefèvre, From damage models to SIR epidemics and cascading failures, Advances in Applied Probability, vol. 41, 2009, pp. 247-269.

[7] P. Jagers, *Branching processes with biological applications*, Wiley London,New York, 1975.

[8] P. Guttorp, *Statistical inference for branching processes*, Wiley, NY, 1991.

[9] J. Kim, Properties of the branching model and the cascading model of the failure propagation of the power network. MS thesis, Electrical and Computer Engineering Department, University of Wisconsin, Madison WI 53706 USA, 2008.

[10] I. Dobson, J. Chen, J.S. Thorp, B.A. Carreras, D.E. Newman, Examining criticality of blackouts in power system models with cascading events, Thirty-fifth Hawaii International Conference on System Sciences, Hawaii, January 2002

[11] I. Dobson, B.A. Carreras, D.E. Newman, A probabilistic loading-dependent model of cascading failure and possible implications for blackouts, Thirty-sixth Hawaii International Conference on System Sciences, Hawaii, January 2003.

[12] I. Dobson, B.A. Carreras, D.E. Newman, Probabilistic load-dependent cascading failure with limited component interactions, IEEE International Symposium on Circuits and Systems, Vancouver Canada, May 2004.

[13] C. Lefèvre, On the outcome of a cascading failure model, Probability in the Engineering and Informational Sciences, vol. 20, Issue 3, 2006, pp. 413-427.

[14] B.A. Carreras, V.E. Lynch, D.E. Newman, I. Dobson, Dynamical and probabilistic approaches to the study of blackout vulnerability of the power transmission grid, Thirty-seventh Hawaii International Conference on System Sciences, Hawaii, January 2004

[15] I. Dobson, B.A. Carreras, V.E. Lynch, B. Nkei, D.E. Newman, Estimating failure propagation in models of cascading blackouts, Probability in the Engineering and Informational Sciences, vol. 19, no. 4, October 2005, pp 475-488.

[16] P. C. Consul, A simple urn model dependent upon predetermined strategy, Sankhyā: The Indian Journal of Statistics, Series B, vol. 36, no. 4, pp. 391-399, 1974.

[17] P. C. Consul, On some models leading to the generalized Poisson distribution, Communications in Statistics - Theory and Methods, vol. 17, no. 2, 1988, pp. 423-442.

[18] P. C. Consul, *Generalized Poisson distributions*, Dekker, NY 1989.
[19] I. Dobson, B.A. Carreras, D.E. Newman, Branching process models for the exponentially increasing portions of cascading failure blackouts, 38th Hawaii International Conference on System Sciences, January 2005, Hawaii.
[20] B.A. Carreras, D.E. Newman, I. Dobson, A.B. Poole, Evidence for self organized criticality in a time series of electric power system blackouts, IEEE Transactions on Circuits and Systems I, vol. 51, no. 9, September 2004, pp. 1733-1740.
[21] I. Dobson, B.A. Carreras, V.E. Lynch, D.E. Newman, Complex systems analysis of series of blackouts: cascading failure, critical points, and self-organization, Chaos, vol. 17, 026103, June 2007.
[22] B.A. Carreras, V.E. Lynch, I. Dobson, D.E. Newman, Critical points and transitions in an electric power transmission model for cascading failure blackouts, Chaos, vol. 12, no. 4, December 2002, pp. 985-994.
[23] B.A. Carreras, V.E. Lynch, I. Dobson, D.E. Newman, Complex dynamics of blackouts in power transmission systems, Chaos, vol. 14, no. 3, September 2004, pp. 643-652.
[24] J. Chen, J.S. Thorp, I. Dobson, Cascading dynamics and mitigation assessment in power system disturbances via a hidden failure model, International Journal of Electrical Power and Energy Systems, vol 27, no 4, May 2005, pp. 318-326.
[25] D.P. Nedic, I. Dobson, D.S. Kirschen, B.A. Carreras, V.E. Lynch, Criticality in a cascading failure blackout model, International Journal of Electrical Power and Energy Systems, vol 28, 2006, pp 627-633.
[26] H. Ren, I. Dobson, Using transmission line outage data to estimate cascading failure propagation in an electric power system, IEEE Transactions on Circuits and Systems Part II, vol. 55, no. 9, Sept. 2008, pp. 927-931.
[27] I. Dobson, J. Kim, K.R. Wierzbicki, Testing branching process estimators of cascading failure with data from a simulation of transmission line outages, *Risk Analysis*, vol. 30, no. 4, 2010, pp. 650 - 662.
[28] K.R. Wierzbicki, I. Dobson, An approach to statistical estimation of cascading failure propagation in blackouts, CRIS, Third International Conference on Critical Infrastructures, Alexandria VA, Sept. 2006.
[29] J. Kim, I. Dobson, Propagation of load shed in cascading line outages simulated by OPA, COMPENG 2010: Complexity in Engineering, Rome Italy, February 2010.
[30] A.D. Barbour, S. Utev, Approximating the Reed-Frost epidemic process, Stochastic Process Applications, vol. 113, May 2004, pp. 173-197.
[31] C. Lefèvre, S. Utev, Branching approximation for the collective epidemic model, Methodology and Computing in Applied Probability, vol. 1, 1999, pp. 211-228.
[32] F. Ball, P. Donnelly, Strong approximations for epidemic models, Stochastic Processes Applications, vol. 55, 1995, pp. 1-21.
[33] Q. Chen, C. Jiang, W. Qiu, J.D. McCalley, Probability models for estimating the probabilities of cascading outages in high-voltage transmission network, IEEE Transactions on Power Systems, vol. 21, no. 3, August 2006, pp. 1423-1431.
[34] R. Adler, S. Daniel, C. Heising, M. Lauby, R. Ludorf, T. White, An IEEE survey of US and Canadian overhead transmission outages at 230 kV and above, IEEE Transactions on Power Delivery, vol. 9, no. 1, Jan. 1994, pp. 21 -39.
[35] M.E.J. Newman, The structure and function of complex networks, SIAM Review, vol. 45, no. 2, 2003, pp. 167256.
[36] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, D-U. Hwanga, Complex networks: structure and dynamics. Physics Reports. 424: 175-308, 2006
[37] D.J. Watts, A simple model of global cascades on random networks, Proceedings of the National Academy of Sciences USA, vol. 99, no. 9, 2002, pp. 5766-5771.
[38] A.E. Motter, Y-C. Lai, Cascade-based attacks on complex networks, Physical Review E, vol 66, no. 6: 065102, 2002.
[39] P. Crucitti, V. Latora, M. Marchiori, Model for cascading failures in complex networks, Physical Review E, 69: 045104(R), 2004.
[40] B. C. Lesieutre, S. Roy, V. Donde, A. Pinar, Power system extreme event screening using graph partitioning, 38th North American Power Symposium, Southern Illinois University Carbondale IL USA, September 2006.
[41] S. Roy, C. Asavathiratham, B. C. Lesieutre, G. C. Verghese, Network models: growth, dynamics, and failure, 34th Hawaii International Conference on System Sciences, Maui, Hawaii, Jan. 2001.
[42] Y. Sun, L. Ma, J. Matthew, S. Zheng, An analytical model for interactive failures, Reliability Engineering and System Safety, vol. 91, 2006, pp. 495-503.
[43] D.V. Lindley, N.D. Singpurwalla, On exchangeable, causal and cascading failures, Statistical Science, 2002, vol. 17, no. 2, pp. 209-219.
[44] IEEE PES CAMS Task Force on Cascading Failure, Initial review of methods for cascading failure analysis in electric power transmission systems, IEEE Power Engineering Society General Meeting, Pittsburgh PA USA, July 2008.
[45] G.L. Greig, Second moment reliability analysis of redundant systems with dependent failures, Reliability Engineering and System Safety, vol. 41, 1993, pp. 57-70.
[46] W. Feller, *An introduction to probability theory and the applications*, vol. 1, New York, 1968, pp. 50-53.

Janghoon Kim (S 05) received the BS degree in electrical power systems from Seoul National University, Seoul, Korea in 2002. He worked for Sewoo Motors from 2002 to 2005. He received the MS degree in electrical engineering from the University of Wisconsin-Madison, USA in 2008. He is currently a graduate student at the University of Wisconsin-Madison. His areas of interest include modeling, simulating and analysis of power system failures.

Ian Dobson (F 06) received the BA degree in Mathematics from Cambridge, England in 1978 and the PhD degree in Electrical Engineering from Cornell University USA in 1989. He worked as a systems analyst for the British firm EASAMS Ltd. from 1978 to 1983. In 1989 he joined the University of Wisconsin-Madison USA, where he is now professor in electrical and computer engineering. His interests are cascading failure risk, complex systems, blackouts, electric power system stability, and nonlinear dynamics.