Properties of the branching model and the cascading model of the failure propagation of the power network

by

Janghoon Kim

A thesis submitted in partial fulfillment of
the requirements for the degree of

Master of Science
(Electrical Engineering)

at the

UNIVERSITY OF WISCONSIN-MADISON
2008

Approved : ______________________________

Ian Dobson, Professor

Date : ______________

Acknowledgments

I gratefully acknowledge support in part from the Power Systems Engineering Research Center (PSERC) and NSF grants ECCS-0606003 and SES-0623985.

Also I gratefully acknowledge support in part from the scholarship of Korean Electric Power Cooperation.

Janghoon Kim

May 01, 2008

Madison, Wisconsin, U.S.A.

Table of Contents

List of Tables

List of Figures

Abstract

Large blackouts can cause considerable damage. For example, in August 2003, the blackout in Northeastern America and Eastern Canada deprived about 50 million people of electricity and cost over 6 billion dollars. However, most large blackouts are not caused by a single big failure. Usually a small failure that does not die out but propagates in the network in a cascade of failures causes a large blackout. So it is useful to establish and investigate models that can explain this phenomenon. The blackout is a complex phenomenon and in a very large network, the modeling of the blackout is not a simple problem. So there exist many various models. We focus on two high-level abstract models, a cascading model and a branching process model. We show quantitatively how the cascading model can be well approximated by the branching process model in many cases. We show aspects of the statistical estimation of parameters of the branching process model from simulated data. In particular, we find an unbiased estimator of the offspring mean and average propagation since the standard estimator is biased when there is a saturation effect. The performance of the estimator is verified by using the estimated parameters to predict the distribution of the number of failures in simulated cascades. These results increase the understanding and application of branching process models to cascading failures.

Chapter 1

Introduction

There are many components of various types in the power network such as generators, transmission lines, loads, transformers and protection and control devices. A failure in the power system means an interruption of the intended operation of a component in the system. It can be a shut down of a generator, breakdown of a fuse, a shorted line, or any abnormal condition of the component that does not provide the right service as it is designed.

A cascading failure is a sequence of dependent failure of some components that weakens a system and incurs recursively additional failures of the remaining components. Ultimately a large portion of the power network can suffer blackout from the failures. Our interest is in the cases in which the cascading failure grows into a large scale blackout.

Recent research shows that North American blackouts that are reportable to NERC happen at an average rate of one every 13 days. Moreover the probability of a large blackout is relatively high [1]. Considering the cost of blackouts, this is a problem.

Regarding this problem, Dobson, et al. [2], [3] show two high level mathematical models of cascading failures, the loading dependent cascading model, CASCADE and the branching process model to analyze the blackout. These models capture the probabilistic aspects of blackouts and give a tool to calculate the probability of large size blackouts. Our research is focused on these two models, the loading dependent cascading model and the branching process model.

Initial work shows that the CASCADE model can capture some aspects of blackout [2]. Also the paper [3] shows that the CASCADE model can be approximated by a branching process in the case of a very large number of components by treating saturations to limits. However, the results of [3] are qualitative and do not quantify the closeness of the approximation or deal with saturation effects.

In this thesis, we focus on accurately quantifying the approximation of two models in the cases that there exist limited number of components and the saturation effect. The thesis expands the previous research to understand the high-level mathematical model of blackouts.

So with better knowledge of the relation of the two models, one can apply the result from one model to another with more ease. By contributing to better understanding of the two models, this thesis provides improved methods to analyze the blackout data, predict the probability of the large blackouts and hence calculate the expectation of loss and risk.

Fitting and analyzing the output data of simulations with the mathematical model is also useful. We find the estimators of parameters. Regarding the branching process model, when the process is governed by the Poisson distribution, we know there is a standard estimator of λ which is effective in the case of infinite components. However, when there are limited components the standard estimator is biased [4]. We suggest a new unbiased estimator for this case. We prove its unbiasedness and compare the performance of new estimator on various cases of the branching Poisson process to that of standard estimator which is unbiased in the infinite component case [5]. A simple simulation of the branching process model is used for this purpose. Also the bounds on the variance of the estimator will be discussed though not perfectly calculated.

1.1 Literature Review

This section consists of two parts. The first part is about the literature about the CASCADE model and the branching process model. Also it reviews the previous research of the approximation of probability models. The second part reviews the research related to estimation in the branching process model.

While the mathematical modeling of blackouts gives the intuition of theoretical concepts of cascading of a failure, there is another approach to understand blackouts using simulations. These simulations are not same as the real world but they are much more complex than our high-level models and model the components and physics of the system. The simulations are the bridges that connect the mathematical model and the real world. Dobson, et al. [6] used a program named as OPA to apply and verify the CASCADE model to the behavior of simulated blackouts. Their OPA assumes the DC load flow and initial failure by random line outages and load variations. And the model simulates the redistribution of overload lines and cascading failures from it. Though their OPA do not reflect operational constraint or other aspects of blackouts, it still represents the probabilistic line outages and overloads. This research shows the CASCADE model is coincident to the results of OPA simulations as the probability distribution of blackout size of model is well fit to the result of simulations. Chen, et al. [7] suggested a hidden failure model that deals with protection systems which will not operate properly in case of needs but is not known its defectiveness until then and showed it roughly follows the NERC data and WSCC system. Nedic , et al. [8] showed the result of the Manchester model simulation that indicates the critical loading at which the mean blackout size increases rapidly and a power law in probability distribution of blackout size. There are other approaches using simulation models to reproduce observed blackouts. Hardiman, et al [9] give the industry grade simulation TRELSS of cascading failure that accounts for several blackout mechanisms. Kosterev, et al. [10] figure out the August 10 1996 WSCC blackout with the EPRI ETMSP program. Chen, et al. [11] shows EAC model fits well to the data of North America. Also there are reports that directly investigate the process of blackouts [12], [13].

1.1.1 Probability models with cascading phenomenon

Cascading is not a phenomenon restricted to the power network. Similar propagation in networks can be seen in several fields. Epidemics are a famous case and mathematical modeling of epidemics are much researched [14], [15]. However the epidemic model is different from cascading failure in the power network since epidemic models assume the dying out or recovery of components during the cascading process while the initial blackout models assume there is no recovery in the short time of a failure propagating in the power network.

The Reed-Frost epidemic process is one of the epidemic models which is similar but not the same as the load dependent cascading model. It assumes that the infection of a component from infected components is governed by a probability. Also it assumes that the infected components can have only one chance of infecting another component. This is different since in the load depending cascading model the failed component gives some load to all other components with probability 1. Barbour and Utev showed the Reed-Frost model is approximated by a branching process in relative closeness in the case of infinite components [16]. Relative closeness means that if the logarithm of the ratio of two functions is within ϵ except for a certain range η, it is in relative closeness with parameters of ϵ, η.

The approximation of the loading dependent cascading model and the branching process model in the power network is also researched as previously described [2], [3]. They showed the approximation is good with an infinite number of components.

This thesis expands these researches. We deal not only with an infinite number of components but also the finite number of components found in the real power network.

1.1.2 Estimation in branching processes

The estimation of offspring distribution in branching processes is a problem even if we have data. It is known that the use of standard estimator in the finite components case produces a bias in the presence of saturation even if the data is generated exactly by the Poisson branching process model [4].

Also the estimation of offspring distributions for a branching process with the real world data is done in several researches [17], [18]. They show the fair description of real world data. Though not directly related to our thesis, there is a research using different models to describe the real world data of blackouts [19].

This thesis is a step to provide a better estimator to the branching process model to enhance these efforts to analyze the blackouts.

1.2 Mathematical definition of the two models

Before providing the proof of approximation of two models, we summarize each model in a detail.

1.2.1 Loading dependent cascade model

In the CASCADE model [20] with n components, we assume the load of each component is loaded as $L_j, j = 1, 2, ..., n$ between L_{min}, L_{max} in the uniform distribution. This is a virtual load concept that roughly summarizes all the factor which affects the loading and failure of the component. When the load is over L_{fail}, it is assumed to fail. The load L_{min} does not necessarily mean zero load in the component. Rather, it means the minimum operating load for the network components.

When there happens some initial disturbances in the network such as the operating errors, the breakdown of some components, the tripping of a line, etc., the model assumes that the initial disturbance load D is added to all the components. If this D makes some components fail then each fail adds additional load P to all the other components. Again, if these P make the remained components fail, then P is added to all the other components. This process goes on until either all components fail or no more component fails. This is the CASCADE model.

We can normalize the CASCADE model by

$$l_j = \frac{L_j - L_{min}}{L_{max} - L_{min}}, \quad p = \frac{P}{L_{max} - L_{min}}, \quad d = \frac{D + L_{max} - L_{fail}}{L_{max} - L_{min}}$$

This changes the load to be distributed between $[0, 1]$. Then the probability distribution of the number of failed components r for given n,d,p is

$$\begin{aligned} C(r, d, p, n) &= \frac{n!}{r!(n-r)!} d(rp + d)^{r-1}(1 - rp - d)^{n-r} \\ & \quad (rp < (1 - d), r < n) \qquad (1.1) \\ C(n, d, p, n) &= 1 - \sum_{r=0}^{n-1} C(r, d, p, n) \qquad (1.2) \\ C(r, d, p, n) &= 0 \ \ \text{(otherwise)} \qquad (1.3) \end{aligned}$$

1.2.2 Branching process model with the saturation

The branching process [3] produces failures in stages starting from some initial failures. In the branching process model with n components, we assume the initial failure of components are generated by some probability distribution called the initial distribution. Then these initially failed components generate the new failed components according to another probability distribution called the offspring distribution. The offspring distribution is the number of failures in the next stage assuming one failure in the previous stage. If there are several failures in a stage they each independently produce failures in the next stage according to the offspring distribution. The generated failures generate new failures until all components fail or the propagation of failures stops.

Suppose that the distribution of an initial failure is the Poisson distribution with a parameter θ and the offspring distribution is a Poisson process with a parameter λ, so

$$P_{initial}\left[k\right] = \frac{\theta^k}{k!}e^{-\theta} \tag{1.4}$$

$$P_{offspring}\left[k\right] = \frac{\lambda^k}{k!}e^{-\lambda} \tag{1.5}$$

Then the distribution of the total number of failures r for given n, λ, θ is

$$B(r,\theta,\lambda,n) = \theta(r\lambda+\theta)^{r-1}\frac{e^{-r\lambda-\theta}}{r!} \qquad (r < n) \tag{1.6}$$

$$B(n,\theta,\lambda,n) = 1 - \sum_{r=0}^{n-1} B(r,\theta,\lambda,n) \tag{1.7}$$

Chapter 2

Analytic approximation of CASCADE model by the branching process model

2.1 Quantifying closeness of models

When the loading dependent cascading model, CASCADE, is given with its parameters, d, p, n, we define the corresponding branching process model as the branching process model with parameters the same n and $\theta = nd, \lambda = np$. Also for a branching model that has parameters θ, λ, n, the corresponding CASCADE model has parameters $d = \frac{\theta}{n}, p = \frac{\lambda}{n}$. For these corresponding models, there are two kinds of approximation.
For $r < n$,

$$R(r, \theta, \lambda, n) = \frac{B(r, \theta, \lambda, n)}{C(r, \theta/n, \lambda/n, n)} \quad (2.1)$$

$$= \frac{(n-r)!n^n e^{-r\lambda-\theta}}{n!(n-r\lambda-\theta)^{n-r}} \quad (2.2)$$

$$A(r, \theta, \lambda, n) = |B(r, \theta, \lambda, n) - C(r, \theta/n, \lambda/n, n)| \quad (2.3)$$

$R(r, \theta, \lambda, n)$, the rational approximation, is the ratio of the probability distributions of total failures by the two models. $A(r, \theta, \lambda, n)$, the absolute approximation, is the absolute difference of them.

For the rational approximation, we have the following results.

Lemma 1 *For some integer a such that* $0 \le r \le a < n, 0 \le \lambda \le 1$, *and* $\theta \le \frac{(1-\lambda)a}{2}$,

$$R(r,\theta,\lambda,n) < \exp\left(\frac{(a(1-\lambda)-\theta)^2}{n-a}\right) \tag{2.4}$$

Lemma 2 *For some integer a such that* $0 \le r \le a < \frac{\frac{2}{3}n-\theta}{\lambda-\frac{1}{3}}$, $\lambda > 1$, *and* $\theta < \frac{n}{2}$,

$$R(r,\theta,\lambda,n) < \exp\left(\frac{(a(1-\lambda)-\theta)^2}{n-a}\right) \tag{2.5}$$

Lemma 3 *For some integer a such that* $0 \le r \le a$, *and* $n - \lambda a - \theta > 0$,

$$R(r,\theta,\lambda,n) > \sqrt{1-\frac{a}{n}} \tag{2.6}$$

From lemmas 1 and 2, the rational approximation of two models is less than 2 where r is less than $\frac{0.7(\sqrt{n}-\theta)}{|1-\lambda|}$. From lemma 3, the rational approximation of two models is greater than $\frac{1}{2}$ when r is less than $\frac{3n}{4}$. Together, we get the conclusion r should be lesser than $\min\{\frac{0.7(\sqrt{n}-\theta)}{|1-\lambda|}, \frac{3n}{4}\}$ to get the approximation between $\frac{1}{2}$ and 2. Usually $0.7\frac{\sqrt{n}-\theta}{|1-\lambda|}$ is smaller than $\frac{3n}{4}$. This explains why the table of exact ratios in next chapter appears to be governed by this bound.

We have shown the two models exhibit a good rational approximation when r is small enough in lemma 1,2. This is not enough to make the approximation of two models close in practice since too large an absolute error will reduce the closeness of the approximation even if the ratio is good. So we investigate the absolute approximation in addition to the rational one.

Lemma 4 *For some integer a such that* $0 \le r \le a$ *and*
(a) If $0 \le \lambda \le 1$,*then* $a < n$ *and* $\theta \le \frac{(1-\lambda)a}{2}$
(b) If $\lambda > 1$, *then* $a < \frac{\frac{2}{3}n-\theta}{\lambda-\frac{1}{3}}$ *and* $\theta < \frac{n}{2}$

(c) $\theta < n - \lambda a$

then

$|B(r,\theta,\lambda,n) - C(r,\theta,\lambda,n)| \leq max\{exp(\frac{(r(1-\lambda)-\theta)^2}{n-r}) - 1, 1 - \sqrt{1-\frac{r}{n}}\}$

The absolute approximation is bounded for large n and small r .

2.2 Proofs for the results

The following section proves lemmas that were stated in previous sections. In addition to the proofs, we will explain the meaning and usage of the lemmas in more detail.

2.2.1 Proof of Lemma 1

For convenience, we state the lemma again.

Lemma 1

For some integer a such that $0 \leq r \leq a < n, 0 \leq \lambda \leq 1, \theta \leq \frac{(1-\lambda)a}{2}$,

$$R(r,\theta,\lambda,n) < \exp\left(\frac{(a(1-\lambda)-\theta)^2}{n-a}\right) \tag{2.7}$$

Proof:

First consider the case $r = 0$.

Define $f(x) = \ln(1-x) + x + x^2$, $f(0) = 0, f'(x) = -\frac{1}{1-x} + 1 + 2x > 0$, for $0 < x < \frac{1}{2}$.

Therefore $f(x) > 0$ and $x + x^2 > -\ln(1-x)$ for $0 < x < \frac{1}{2}$. Hence

$$\begin{aligned} \ln R(r,\theta,\lambda,n) &= -\theta - n\ln(1-\frac{\theta}{n}) \\ &< -\theta + n(\frac{\theta}{n} + \frac{\theta^2}{n^2}) \\ &< \frac{\theta^2}{n} \\ &< \frac{(a(1-\lambda)-\theta)^2}{n-a} \end{aligned}$$

Now consider the case $r \geq 1$, we use the Stirling approximation inequality [21],

$$\sqrt{2n\pi} n^n e^{-n+\frac{1}{12n+1}} < n! < \sqrt{2n\pi} n^n e^{-n+\frac{1}{12n}} \tag{2.8}$$

then by substituting for $n!$ and $(n-r)!$ in (2.2) we get

$$R(r,\theta,\lambda,n) < R_{\max}(r,\theta,\lambda,n)$$

where

$$\begin{aligned} R_{\max}(r,\theta,\lambda,n) &= \frac{\sqrt{2(n-r)\pi}\,(n-r)^{n-r}e^{-n+r+\frac{1}{12(n-r)}}}{\sqrt{2n\pi}\,n^n e^{-n+\frac{1}{12n+1}}} \\ &\times\ n^n(n-\lambda r-\theta)^{r-n}e^{-\lambda r-\theta} \\ &= \frac{\sqrt{(n-r)}\,(n-r)^{n-r}e^{r+\frac{1}{12(n-r)}}}{\sqrt{n}\,e^{\frac{1}{12n+1}}} \\ &\times\ (n-\lambda r-\theta)^{r-n}e^{-\lambda r-\theta} \end{aligned}$$

As $r \geq 1$, $n \geq 2$. This implies

$$0.5\ln(1-\frac{r}{n}) + \frac{r+\frac{1}{12}}{12(n-r)(n+\frac{1}{12})} < -\frac{r}{2n} + \frac{r+\frac{1}{12}}{12n(n-r)} < 0$$

So

$$\begin{aligned}\ln R_{\max}(r,\theta,\lambda,n) &= 0.5\ln(1-\frac{r}{n}+\frac{r+\frac{1}{12}}{12(n-r)(n+\frac{1}{12})}) \\ &+ (1-\lambda)r-\theta-(n-r)\ln(1+\frac{(1-\lambda)r-\theta}{n-r}) \\ &< (1-\lambda)r-\theta-(n-r)\ln(1+\frac{(1-\lambda)r-\theta}{n-r})\end{aligned}$$

Define the function $f_2(x) = \ln(1+x) - x + \frac{x^2}{2}$.

$$\begin{aligned}f_2(0) &= 0 \\ f_2'(x) &= \frac{1}{1+x} - 1 + x > 0 \;\; for \;\; x > 0\end{aligned}$$

Hence $\ln(1+x) \geq x - \frac{x^2}{2}$ for $x \geq 0$ and

$$\ln R_{\max} \leq \frac{((1-\lambda)r-\theta)^2}{2(n-r)} \tag{2.9}$$

$$< \frac{(a(1-\lambda)-\theta)^2}{n-a} \tag{2.10}$$

2.2.2 Proof of Lemma 2

Lemma 2

For some integer a such that $0 \leq r \leq a < \frac{\frac{2}{3}n-\theta}{\lambda-\frac{1}{3}}$,$\lambda > 1$,$\theta < \frac{n}{2}$,

$$R(r,\theta,\lambda,n) < \exp\left(\frac{(a(1-\lambda)-\theta)^2}{n-a}\right) \tag{2.11}$$

Proof:

Consider the case $r = 0, 2\theta < n$.

Using the fact that $x + x^2 > -\ln(1-x)$ and $a(1-\lambda) - \theta < -\theta < 0$, we get

$$\begin{aligned}\ln R(r,\theta,\lambda,n) &= -\theta - n\ln(1-\frac{\theta}{n}) \\ &< \frac{\theta^2}{n} \\ &< \frac{(a(1-\lambda)-\theta)^2}{n-a}\end{aligned}$$

Now consider the case $r \geq 1$.

$$\begin{aligned}\ln R(r,\theta,\lambda,n) &< \ln R_{\max}(r,\theta,\lambda,n) \\ &< (1-\lambda)r - \theta - (n-r)\ln(1+\frac{(1-\lambda)r-\theta}{n-r})\end{aligned}$$

Define the function $g(x) = \ln(1+x) - x + x^2$.

$$\begin{aligned}g(0) &= 0 \\ g'(x) &= \frac{1}{1+x} - 1 + 2x \\ &= \frac{x(2x+1)}{1+x}\end{aligned}$$

Since $g'(x) \geq 0$ $(-\frac{2}{3} < x \leq -\frac{1}{2}), g'(x) < 0$ $(-\frac{1}{2} < x < 0), g(0) = 0$ $g(-\frac{2}{3}) > 0$, $g(x) > 0$ for $-\frac{2}{3} < x < 0$. We note that as $\lambda > 1, \frac{\frac{2}{3}n-\theta}{\lambda-\frac{1}{3}} < \frac{\frac{2}{3}n}{\lambda-\frac{1}{3}} < n$ and $(1-\lambda)r - \theta < 0$. Let $\alpha = \frac{(1-\lambda)a-\theta}{n-a}$. As $1 \leq r \leq a, \frac{(1-\lambda)a-\theta}{n-a} \leq \frac{(1-\lambda)a-\theta}{n-r} \leq \frac{(1-\lambda)r-\theta}{n-r}$. So $-\frac{2}{3} < \alpha \leq x < 0$.

Also $0 < n-a \leq n-r$, $(1-\lambda)a - \theta \leq (1-\lambda)r - \theta < 0$. So we get

$$\begin{aligned}\ln R_{\max} &< (n-r)\frac{((1-\lambda)r-\theta)^2}{(n-r)^2} \\ &\leq \frac{((1-\lambda)a-\theta)^2}{n-a}\end{aligned}$$

2.2.3 Proof of Lemma 3

Lemma 3

For some integer a such that $r \le a$,$n - \lambda a - \theta > 0$,

$$R(r, \theta, \lambda, n) > \sqrt{1 - \frac{a}{n}} \tag{2.12}$$

Proof: Consider the case $r = 0$.

$$\begin{aligned} \ln R(r, \theta, \lambda, n) &= -\theta - n \ln(1 - \frac{\theta}{n}) \\ &> 0 \\ &\geq 0.5 \ln(1 - \frac{a}{n}) \end{aligned}$$

Now consider the case $1 \le r \le a$.

By using the Stirling's approximation (2.8), we get

$$\begin{aligned} \ln R(r, \theta, \lambda, n) &> \ln \left\{ \frac{\sqrt{2(n-r)\pi}(n-r)^{n-r} e^{-n+r+\frac{1}{12(n-r)+1}}}{\sqrt{2n\pi} n^n e^{-n+\frac{1}{12n}}} e^{-\lambda r - \theta} \frac{(n - \lambda r - \theta)^r}{(1 - \frac{\lambda r + \theta}{n})^n} \right\} \\ &= 0.5 \ln(1 - \frac{r}{n}) + \frac{1}{12(n-r)+1} - \frac{1}{12n} \\ &\quad + ((1-\lambda) r - \theta) - (n - r) \ln \left(1 + \frac{(1-\lambda) r - \theta}{n - r} \right) \end{aligned}$$

Define the function $h(x) = \ln(1 + x) - x$.

$$\begin{aligned} h(0) &= 0 \\ h'(x) &= \frac{1}{1+x} - 1 \end{aligned}$$

For $x > 0$ $i'(x) < 0$ and $-1 < x < 0$ $h'(x) > 0$. Therefore $h(x) \leq 0$ for $-1 < x$. Let $x = \frac{(1-\lambda)r-\theta}{n-r}$. Then, since $(1-\lambda)a - \theta > -n + a$, $x > -1$.

$$((1-\lambda)r-\theta) - (n-r)\ln(1 + \frac{(1-\lambda)r-\theta}{n-r}) \geq 0.$$

So

$$\begin{aligned}\ln R(r,\theta,\lambda,n) &> 0.5\ln(1-\frac{a}{n}) + \frac{1}{12(n-a)+1} - \frac{1}{12n} \\ &> 0.5\ln(1-\frac{a}{n})\end{aligned}$$

2.2.4 Proof of Lemma 4

Proof:

As the probability of an event never exceeds 1, let think of two number $0 < a \leq 1, 0 \leq b \leq 1$.

$$\begin{aligned}|a-b| &= |a(1-\frac{b}{a})| \\ &\leq |1-\frac{b}{a}| \\ &\leq \max\{|1-\min\{\frac{b}{a}\}|, |\max\{\frac{b}{a}\}-1|\}\end{aligned}$$

Now substitute $C(r,\theta,\lambda,r)$ and $B(r,\theta,\lambda,r)$ for a and b respectively. Note that from conditions of lemma 4, $C(r,\theta,\lambda,r) \neq 0$. With lemma 1,2,3, we get the result.

Chapter 3

Practical approximation of two model

We proved that the branching process model is approximated to the CASCADE model well within certain bounds in the previous chapter. In this chapter, we evaluate the approximation of two models numerically. While the theoretical analysis gives the intuition for the reason of good approximation, the bounds suggested by it are actually a little looser than the true bounds. So we calculate the numerical approximation of two models for various cases and verify the previous proof. Also by analyzing this numerical data, we will give approximations of the true bounds.

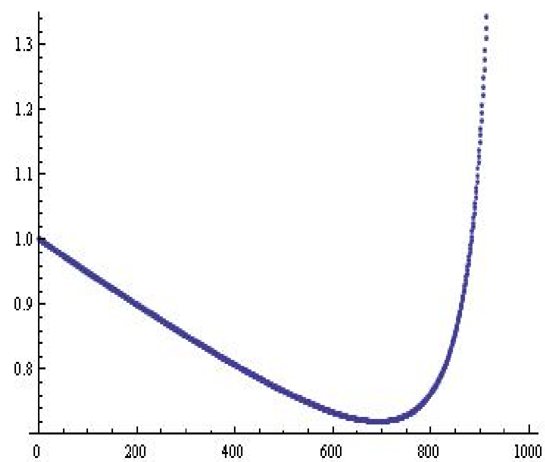

Figure 3.1: Typical ratio $R(r, \theta, \lambda, n)$ when $n = 1000, \theta = 1, \lambda = 0.98$

Figure 3.1 shows one typical case of the rational approximation. We note that the theoretical upper bound on r on $R(r, \theta, \lambda, n)$ proved in the Lemma 1,2 is much lower than the actual upper bound on r. Practically the approximation is lesser than 2 and greater than half until much larger r than the number that the lemma 1,2 guarantee.

We calculated the ratio of two model when $\theta = 0.5, 1, 2,\ \lambda = 0.2, 0.4, 0.6, ..., 1.8, n = 100, 1000, 10000$. br means the minimum r such that the $R(r, \theta, \lambda, n)$ goes greater than two or less than half.

Table 3.1: The value of r where the approximation of the branching process model and the CASCADE model goes greater than two or less than half

n	θ	λ	br	n	θ	λ	br	n	θ	λ	br
100	0.5	0.2	16	100	0.5	0.4	21	100	0.5	0.6	31
100	0.5	0.8	56	100	0.5	1.0	76	100	0.5	1.2	47
100	0.5	1.4	26	100	0.5	1.6	18	100	0.5	1.8	14
100	1	0.2	17	100	1	0.4	22	100	1	0.6	32
100	1	0.8	58	100	1	1	77	100	1	1.2	45
100	1	1.4	25	100	1	1.6	17	100	1	1.8	13
100	2	0.2	18	100	2	0.4	24	100	2	0.6	34
100	2	0.8	62	100	2	1	80	100	2	1.2	41
100	2	1.4	22	100	1	1.6	15	100	2	1.8	12
1000	0.5	0.2	48	1000	0.5	0.4	64	1000	0.5	0.6	95
1000	0.5	0.8	186	1000	0.5	1.0	751	1000	0.5	1.2	176
1000	0.5	1.4	90	1000	0.5	1.6	60	1000	0.5	1.8	46
1000	1	0.2	49	1000	1	0.4	64	1000	1	0.6	96
1000	1	0.8	188	1000	1	1	752	1000	1	1.2	173
1000	1	1.4	89	1000	1	1.6	60	1000	1	1.8	45
1000	2	0.2	50	1000	2	0.4	66	1000	2	0.6	98
1000	2	0.8	193	1000	2	1	755	1000	2	1.2	169
1000	2	1.4	86	1000	1	1.6	58	1000	2	1.8	44
10000	0.5	0.2	149	10000	0.5	0.4	198	10000	0.5	0.6	296
10000	0.5	0.8	589	10000	0.5	1.0	7501	10000	0.5	1.2	579
10000	0.5	1.4	291	10000	0.5	1.6	195	10000	0.5	1.8	148
10000	1	0.2	149	10000	1	0.4	199	10000	1	0.6	297
10000	1	0.8	591	10000	1	1	7502	10000	1	1.2	577
10000	1	1.4	290	10000	1	1.6	194	10000	1	1.8	146
10000	2	0.2	150	10000	2	0.4	200	10000	2	0.6	300
10000	2	0.8	596	10000	2	1	7505	10000	2	1.2	572
10000	2	1.4	287	10000	1	1.6	192	10000	2	1.8	144

Though I defined that br is the point at which the ratio is either two or half, except the case that $\lambda = 1$, the ratio goes greater than two at br and not less than half.

As indicated in the graph, the branching process model exhibits smaller probability at small r than the CASCADE model does. And for r greater than the certain number which is different in every case but approximately $\frac{\sqrt{n}+\theta}{|1-\lambda|}$ in most cases, the ratio of branching process model to CASCADE becomes larger than 1 and increases rapidly above that number. However for these large r, both probabilities of CASCADE and the branching process model are very small.

Chapter 4

An improved estimator of branching process

4.1 The previous estimator and definitions related to the branching process

When the data set is given, the method of fitting it to the branching process model is a problem. The method to fit a blackout and line failures to the branching process model was already dealt in statistical estimation of cascading blackout size and propagation with branching process by Kevin R. Wierzbicki [4]. He provided the standard estimator of λ, θ. However, the estimator in that article is biased. In this thesis, I suggest a new estimator that is unbiased.

Before explaining the new estimator, I will describe the branching process and define a few notations that will be used to express it. In a branching process, there are initial failures at the first stage. They can have general initial distribution Z_i of finite mean and variance. I define $\lambda_0 = E[Z_i]$.

Each failure of the initial failures generates offspring failures in the second stage independently. These failures generate failures in the next stage again independently until all die out or saturation occurs. The estimator is a function to get the mean of this offspring distribution. I assume that the offspring distribution does not change during the stages. I name this offspring function Z and $\lambda = E[Z]$. The function can be a general function, too.

With these notations, I will define the new estimator and show how it works. I define the number of failures of each stage as $Z_0, Z_1, Z_2....$ The subscript of Z is the stage number. As there can be several failure samples, we indicate the sample number k by a superscript (k).

4.2 Yanev's variance estimation for fixed number of stage and its improvement

Before explaining the new estimator, I review a previous work related to this problem. Yanev [5] gives a proof for the number of stages t and the number of samples K both tending to infinity. Here we sketch his proof that is adapted and reduced to the case such that K tends to infinity and there are a fixed number of stages t. The estimator for a fixed number of stages t is

$$\widehat{\lambda}_t = \frac{\sum_{k=1}^{K} \left(Z_1^{(k)} + Z_2^{(k)} + ... + Z_t^{(k)} \right)}{\sum_{k=1}^{K} \left(Z_0^{(k)} + Z_1^{(k)} + ... + Z_{t-1}^{(k)} \right)} \tag{4.1}$$

Yanev assumes a case such that initial failure distribution is $Z_i = 1$ constant. So in his paper there is no general initial distribution of failures. We modify his work to reflect the case such that initial failure distribution function is general.

Let

$$w_i^{(k)} = Z_{i+1}^{(k)} - \lambda Z_i^{(k)} \tag{4.2}$$

$$Var[w_0^{(k)} | Z_0 = 1] = Var[Z] = \sigma^2 \tag{4.3}$$

It is sometimes convenient to drop the superscript (k). And as $\lambda = E[Z]$, and by definition of offspring distribution,

$$\begin{aligned} Ew_i &= E[E[w_i|Z_i]] = 0 && (4.4) \\ Ew_i^2 &= E[E[w_i^2|Z_i]] = E[\sigma^2 Z_i] = \sigma^2 \lambda_0 \lambda^i && (4.5) \\ Ew_i w_j &= E[E[(Z_{i+1} - \lambda Z_i)(Z_{j+1} - \lambda Z_j)|Z_j]] \\ &= E[(Z_{i+1} - \lambda Z_i) Z_{j+1}] \\ &= E[E[(Z_{i+1} - \lambda Z_i) Z_{j+1}|Z_i, Z_{j+1}]] \\ &= 0 \qquad \text{where } i > j. && (4.6) \end{aligned}$$

It follows that

$$E\left(\sum_{i=0}^{t-1} w_i\right) = 0 \tag{4.7}$$

and

$$E\left[\left(\sum_{i=0}^{t-1} w_i\right)^2\right] = \sigma^2 \lambda_0 \Lambda_{t-1} \tag{4.8}$$

where

$$\Lambda_{t-1} = 1 + \lambda + ... + \lambda^{t-1} \tag{4.9}$$

Now let

$$Y_t^{(k)} = Z_1^{(k)} + Z_2^{(k)} + ... + Z_t^{(k)} \tag{4.10}$$

Then

$$\widehat{\lambda}_t - \lambda = \frac{\frac{1}{K}\sum_{k=1}^{K}\sum_{i=0}^{t-1} w_i^{(k)}}{\frac{1}{K}\sum_{k=1}^{K} Y_{t-1}^{(k)}} \tag{4.11}$$

Since

$$\frac{1}{K}\sum_{k=1}^{K} Y_{t-1}^{(k)} \to EY_{t-1} = \lambda_0 \Lambda_{t-1} \tag{4.12}$$

a.s. as $K \to \infty$, we have from (4.4) that $E(\widehat{\lambda}_t - \lambda) \to 0$ as $K \to \infty$ so that $\widehat{\lambda}_t$ is asymptotically unbiased.

Moreover,

$$\frac{\sqrt{K\lambda_0\Lambda_{t-1}}}{\sigma}(\widehat{\lambda}_t - \lambda) = \frac{\lambda_0\Lambda_{t-1}}{\frac{1}{K}\sum_{k=1}^K Y_{t-1}^{(k)}} \frac{1}{\sigma\sqrt{K\lambda_0\Lambda_{t-1}}} \sum_{k=1}^{K}\sum_{i=0}^{t-1} w_i^{(k)} \tag{4.13}$$

and then (4.12),(4.8) and the CLT imply that

$$\frac{\sqrt{K\lambda_0\Lambda_{t-1}}}{\sigma}(\widehat{\lambda}_t - \lambda) \to N(0,1) \tag{4.14}$$

in probability as $K \to \infty$. Equivalently,

$$\widehat{\lambda}_t \to N(\lambda, \frac{\sigma^2}{K\lambda_0\Lambda_{t-1}}) \tag{4.15}$$

in probability as $K \to \infty$.

4.2.1 Poisson Offspring Distribution

When the offspring distribution is a Poisson distribution, $\sigma^2 = \lambda$. And (4.15) becomes

$$\widehat{\lambda}_t \to N(\lambda, \frac{\lambda}{K\lambda_0\Lambda_{t-1}}) \tag{4.16}$$

in probability as $K \to \infty$.

4.3 New estimator for Poisson Offspring and Initial Distribution with saturation

As previously described, the unbiased estimator in the case with an infinite number of components is researched by Yanev. However, in the practical world, this case is unlikely. The number of components in the network is limited, so saturation applies. Kevin Wierzbicki showed [4] that in the case that saturation applies, the standard estimator of mean for the Poisson offspring distribution is biased and shows a tendency to underestimate the true value. In this thesis, I show the reason for underestimating and suggest the new unbiased estimator.

The proof that the standard estimator $\widehat{\lambda}_n$ is asymptotically unbiased when there is no saturation relies on the fact that $E[Z_{i+1}] = \lambda E[Z_i]$ [5]. When there is saturation, $\widehat{\lambda}_n$ asymptotically underestimates λ because $E[Z_{i+1}] < \lambda E[Z_i]$. The following shows the reason why $E[Z_{i+1}] < \lambda E[Z_i]$.

$$
\begin{aligned}
E[Z_{i+1}] &= E[E[Z_{i+1}|Y_i, Z_i]] \\
&= E\Bigg[\sum_{r=1}^{S-Y_i-1} r\frac{(Z_i\lambda)^r}{r!}e^{-Z_i\lambda} \\
&\qquad\qquad +(S-Y_i)\sum_{r=S-Y_i}^{\infty}\frac{(Z_i\lambda)^r}{r!}e^{-Z_i\lambda}\Bigg] \\
&= E\Bigg[\sum_{r=1}^{\infty} r\frac{(Z_i\lambda)^r}{r!}e^{-Z_i\lambda} \\
&\qquad\qquad -\sum_{r=S-Y_i}^{\infty}(r-S+Y_i)\frac{(Z_i\lambda)^r}{r!}e^{-Z_i\lambda}\Bigg] \\
&< \lambda E\Bigg[Z_i\sum_{r=0}^{\infty}\frac{(Z_i\lambda)^r}{r!}e^{-Z_i\lambda}\Bigg] \\
&= \lambda E[Z_i]
\end{aligned}
$$

So the standard estimator is biased. To compensate this biasness, I suggest the new estimator. Before showing it, I define some more notions.

$$s(k,S) = \max\{\, n \mid Y_n^{(k)} < S \ \text{ and } \ Z_{n-1}^{(k)} > 0 \,\} \tag{4.17}$$

k is the number of each sample and S is the size of saturation. $s(k,S)$ means the number of stage where it is not saturated and the previous stage is not zero. So failures didn't die out in the previous stage. It is either the number of stage just before saturation or the number of stage where failures die out.

The new estimator I suggest is

$$\widehat{\lambda}_s = \frac{\sum_{k=1}^{K}\left(Z_1^{(k)} + Z_2^{(k)} + \dots + Z_{s(k,S)}^{(k)}\right)}{\sum_{k=1}^{K}\left(Z_0^{(k)} + Z_1^{(k)} + \dots + Z_{s(k,S-1)-1}^{(k)}\right)} \tag{4.18}$$

$$= \frac{\sum_{k=1}^{K}\left(Y_{s(k,S)}^{(k)} - Z_0^{(k)}\right)}{\sum_{k=1}^{K} Y_{s(k,S-1)-1}^{(k)}} \tag{4.19}$$

$$\tag{4.20}$$

4.4 Unbiasness of $\widehat{\lambda}_s$

We continue to assume that the branching process has Poisson initial failures with mean θ and a Poisson offspring distribution with mean λ.

To show that $\widehat{\lambda}_s$ is asymptotically unbiased, rewrite (4.20) as

$$\widehat{\lambda}_s = \frac{\frac{1}{K}\sum_{k=1}^{K}\sum_{i=0}^{S-3} Z_{i+1}^{(k)} I[Y_{i+1}^{(k)} < S]}{\frac{1}{K}\sum_{k=1}^{K}\sum_{i=0}^{S-3} Z_{i}^{(k)} I[Y_{i+1}^{(k)} < S-1]}$$

Let

$$w_i^{(k)} = Z_{i+1}^{(k)} I[Y_{i+1}^{(k)} < S] - \lambda Z_i^{(k)} I[Y_{i+1}^{(k)} < S-1]$$

It is sometimes convenient to omit the superscript and to write w_i for $w_i^{(k)}$. Then

$$\widehat{\lambda}_s - \lambda = \frac{\frac{1}{K}\sum_{k=1}^{K}\sum_{i=0}^{S-3} w_i^{(k)}}{\frac{1}{K}\sum_{k=1}^{K} Y_{s(k,S-1)-1}^{(k)}} \tag{4.21}$$

As each k, $Y_{s(k,S-1)-1}^{(k)}$ is bounded by S and has finite mean and variance. Moreover, $Y_{s(k,S-1)-1}^{(k)}$, $k = 1, 2, ..., K$ are independent and the strong law of large numbers implies that the denominator of (4.21) tends almost surely to a constant.

Therefore to prove that $E(\widehat{\lambda}_s - \lambda) \to 0$ almost surely and $\widehat{\lambda}_s$ is asymptotically unbiased, it is sufficient to show that $Ew_i^{(k)} = 0$ for $i = 0, 1, 2, ..., S-3$. And $Ew_i^{(k)} = 0$ follows from

$$\begin{aligned}
&E[Z_{i+1}I[Y_{i+1} < S]] \\
&\quad = E[E[Z_{i+1}I[Z_{i+1} < S - Y_i]|Y_i, Z_i]] \\
&\quad = E\Big[\sum_{m=1}^{S-Y_i-1} m\frac{(Z_i\lambda)^m}{m!}e^{-Z_i\lambda}\Big] \\
&\quad = \lambda E\Big[\sum_{m=0}^{S-Y_i-2} Z_i\frac{(Z_i\lambda)^m}{m!}e^{-Z_i\lambda}\Big] \\
&\quad = \lambda E\big[E[Z_iI[Y_i + Z_{i+1} < S - 1]|Y_i, Z_i]\big] \\
&\quad = \lambda E[Z_iI[Y_{i+1} < S - 1]].
\end{aligned}$$

So the new estimator is unbiased and asymptotically unbiased.

Chapter 5

Analysis of theoretical bounds of variance of the new estimator

5.1 Convergence of variance of $\widehat{\lambda}_s$

We derive the asymptotic variance of $\widehat{\lambda}_s$ in the subcritical case of $\lambda < 1$ and when saturation is neglected by letting $S \to \infty$. When $\lambda < 1$, the branching process dies out with $Z_i^{(k)} \to 0$ as $i \to \infty$ almost surely and $Y_n^{(k)} \to Y_\infty^{(k)}$ as $n \to \infty$ almost surely. Hence, the Harris estimator $\widehat{\lambda}_n \to \widehat{\lambda}_\infty$ as $n \to \infty$, where

$$\widehat{\lambda}_\infty = \frac{\sum_{k=1}^{K} \left(Y_\infty^{(k)} - Z_0^{(k)} \right)}{\sum_{k=1}^{K} Y_\infty^{(k)}}$$

Moreover, for $\lambda < 1$, our estimator $\widehat{\lambda}_s \to \widehat{\lambda}_\infty$ as $S \to \infty$. From (4.16), the variance of $\widehat{\lambda}_\infty$ as $K \to \infty$ is

$$\sigma^2(\widehat{\lambda}_\infty) = \frac{\lambda(1-\lambda)(1-e^{-\theta})}{K\theta} \tag{5.1}$$

Thus (5.1) gives the asymptotic variance of $\widehat{\lambda}_s$ as $K \to \infty$ and $S \to \infty$ for $\lambda < 1$. For example, for $\theta = 1$, the maximum asymptotic variance occurs for $\lambda = 0.5$ and the asymptotic standard deviation from (5.1) becomes $\sigma(\widehat{\lambda}_\infty) = 0.40/\sqrt{K}$.

5.2 Bounds of variance of λ_s

We showed the convergence of variance. In this chapter, we investigate the variance deeper.

We have

$$\sqrt{K}(\widehat{\lambda}_s - \lambda) = \frac{E[Y_{s(k,S-1)-1}]}{\frac{1}{K}\sum_{k=1}^{K} Y_{s(k,S-1)-1}^{(k)}} \frac{1}{\sqrt{K}E[Y_{s(k,S-1)-1}]} \sum_{k=1}^{K}\sum_{i=0}^{S-3} w_i^{(k)} \tag{5.2}$$

Then (5.2) and the CLT imply that

$$\widehat{\lambda}_s \to N\left(\lambda, A^2\right) \tag{5.3}$$

in probability as $K \to \infty$, where

$$A^2 = \frac{1}{K(E[Y_{s(k,S-1)-1}])^2} E\left[\left(\sum_{i=0}^{S-3} w_i\right)^2\right] \tag{5.4}$$

Now we analyze the bound of this A^2.

$$A^2 = \frac{1}{K(E[Y_{s(k,S-1)-1}])^2} \sum_{i=0}^{S-3} E[w_i^2] \tag{5.5}$$

$$\begin{aligned} E[w_i^2] &= E[(Z_{i+1} - \lambda Z_i)^2 I[Y_{i+1} < S-1] + \\ &\quad E[Z_{i+1}^2 I[Y_{i+1} = S-1]] \\ &\leq E[(Z_{i+1} - \lambda Z_i)^2] + (S-i-1)^2 P[Y_{i+1} = S-1] \\ &= \theta\sigma^2\lambda^i + (S-i-1)^2 P[Y_{i+1} = S-1] \end{aligned} \tag{5.6}$$

We need to know the minimum of $(E[Y_{s(k,S-1)-1}])^2$ and the maximum of $\sum_{i=0}^{S-3} E[w_i^2]$ to get bound of variance. If the distribution is known and can be calculated, we can estimate the bound of the variance from (5.6). We expand the analysis of (5.6) for the case that the initial distribution is a Poisson distribution.

1) $\sum_{i=0}^{S-3} E[w_i^2]$

$$\sum_{i=0}^{S-3} E[w_i^2] \leq \sum_{i=0}^{S-3} \theta\sigma^2\lambda^i + (S-i-1)^2 P[Y_{i+1} = S-1] \tag{5.7}$$
$$< \frac{\theta\sigma^2}{1-\lambda} + (S-1)^2 B(S-1, \theta, \lambda, S) \tag{5.8}$$

$$\begin{aligned} B(r, \theta, \lambda, n) &= \theta(r\lambda+\theta)^{r-1}\frac{e^{-r\lambda-\theta}}{r!} \\ &< \theta(r\lambda+\theta)^{r-1}\frac{e^{-r\lambda-\theta}}{\sqrt{2\pi r}r^r e^{-r+\frac{1}{12r+1}}} \\ &< \theta(\lambda+\frac{\theta}{r})^r\frac{e^{(1-\lambda)r-\theta}}{\sqrt{2\pi r}(r\lambda+\theta)} \\ &= \theta(\lambda e^{(1-\lambda)})^r(1+\frac{\theta}{\lambda r})^r\frac{e^{-\theta}}{\sqrt{2\pi r}(r\lambda+\theta)} \\ &< \theta(\lambda e^{(1-\lambda)})^r\frac{e^{\theta(1/\lambda-1)}}{\sqrt{2\pi r}(r\lambda+\theta)} \end{aligned}$$

So $(S-1)^2 B(S-1, \theta, \lambda, S)$ is proportional to $\sqrt{S-1}(\lambda e^{1-\lambda})^{S-1}$. Since $\lambda e^{1-\lambda} < 1$ for $\lambda < 1$, this is a decreasing function of $S-1$ for large enough S. It is directly calculated that $(S-1)^2 B(S-1, \theta, \lambda, S)$ is lesser than 0.005 for $S > 100, \lambda \leq 0.65$ which is neglectable compared to $\frac{\theta\sigma^2}{1-\lambda}$. Actually, the above bound is larger than the true value we got from direct computer calculations and would be able to be smaller values. However still it suggests that $(S-1)^2 B(S-1, \theta, \lambda, S)$ rapidly becomes small compared to $\frac{\theta\sigma^2}{1-\lambda}$ when λ is small.

As σ is the variance of Poisson offspring distribution, by neglecting the second term in

(5.8),

$$\begin{aligned}\sigma^2 &= E[(Z_1 - \lambda Z_0)^2 | Z_0 = 1] \\ &= E[Z_1^2] - 2\lambda E[Z_1] + \lambda^2 \\ &= Var[Z_1] + E[Z_1]^2 - 2\lambda^2 + \lambda^2 \\ &= \lambda\end{aligned}$$

So

$$\begin{aligned}\sum_{i=0}^{S-3} E[w_i^2] &\leq \frac{\theta\sigma^2}{1-\lambda} \\ &= \frac{\theta\lambda}{1-\lambda}\end{aligned}$$

2) $\frac{1}{(E[Y_{s(k,S-1)-1}])^2}$

$$\begin{aligned}E[Y_{s(k,S-1)-1}] &\geq \sum_{1}^{S-2} rB(r,\theta,\lambda,S) \\ &= E[r|S=\infty] - \sum_{S-1}^{\infty} rB(r,\theta,\lambda,S) \\ &> \frac{\theta}{1-\lambda} - \theta\sqrt{S-1}(\lambda e^{(1-\lambda)})^{S-1}\frac{e^{\theta(1/\lambda-1)}}{\sqrt{2\pi}((S-1)\lambda+\theta)}\end{aligned}$$

The second term is also a decreasing function of S after some large S. So we can approximate $E[\Lambda_s]$ by $\frac{\theta}{1-\lambda}$ in that case. $\theta\sqrt{S-1}(\lambda e^{(1-\lambda)})^{S-1}\frac{e^{\theta(1/\lambda-1)}}{\sqrt{2\pi}((S-1)\lambda+\theta)}$ shows smaller value than 0.0005 for $S > 200, \lambda < 0.65$. In this case, we can neglect $\theta\sqrt{S-1}(\lambda e^{(1-\lambda)})^{S-1}\frac{e^{\theta(1/\lambda-1)}}{\sqrt{2\pi}((S-1)\lambda+\theta)}$.

With this approximation

$$A^2 = \frac{\lambda(1-\lambda)}{K\theta}$$

As the maximum of $\lambda(1-\lambda)$ is 0.25, maximum of standard deviation of estimator A is approximated to the value $\frac{0.5}{\sqrt{K\theta}}$. This shows a clearly similar result to that of Wierzbicki's empirical result of $\frac{0.5}{\sqrt{K}}$ [4].

3)When S is small, λ is large. In this case, the above approximation is not good. We should use the direct calculation of sum of $rg(r,\theta,\lambda,S)$ for the lower bound of $E[Y_{s(k,S-1)-1}]$. Also we should calculate the value of $(S-1)^2 g(S-1,\theta,\lambda,S)$ directly to get the upper bound of $\sum_{i=0}^{S-3} E[w_i^2]$. Further research may suggest a better bound for this case.

Chapter 6

Performance results of the new estimator

To augment these asymptotic results, the estimator $\widehat{\lambda}_s$ is tested on the saturating branching process with $\theta = 1$ and $0 < \lambda < 2$. The worst case bias and standard deviation of $\widehat{\lambda}_s$ are determined numerically from 1000 cascades with nonzero failures and the results are shown in Table 6.1. The asymptotic variance (5.1) and Table 6.1 can be used to estimate the number of cascades K needed to obtain a given standard deviation in $\widehat{\lambda}_s$.

Table 6.1: Bias and standard deviation of $\widehat{\lambda}_s$ on saturating branching process with $\theta = 1$

number of runs K	saturation S	bias $\max\limits_{0<\lambda<2} \lvert\mu(\widehat{\lambda}_s) - \lambda\rvert$	standard deviation $\max\limits_{0<\lambda<2} \sigma(\widehat{\lambda}_s)$
10	20	0.035	$0.28 = 0.87/\sqrt{K}$
20	20	0.018	$0.18 = 0.80/\sqrt{K}$
50	20	0.008	$0.11 = 0.78/\sqrt{K}$
200	20	0.004	$0.055 = 0.77/\sqrt{K}$
10	100	0.050	$0.16 = 0.57/\sqrt{K}$
20	100	0.027	$0.092 = 0.41/\sqrt{K}$
50	100	0.010	$0.057 = 0.40/\sqrt{K}$
200	100	0.003	$0.029 = 0.41/\sqrt{K}$

As S increases, the empirical variance shows clear coincidence with the suggested approximated bound of $\frac{0.4}{\sqrt{K}}$. Though this is not a rigorous proof, it supports the previous theoretical analysis.

Chapter 7

Conclusion and Future Work

In this thesis, the approximation of a loading dependent model of cascading failures by a branching process model is analyzed. Bounds on the region of good approximation are given. We have provided numerical evidence of the actual approximation of the two models, too. The actual approximation supports the theoretical analysis and can be used to determine the maximum number of failed components for which the approximation is good. Except for the cases of extreme values of the parameters λ, θ, the ratio of the probability of r components failed is less than two and greater than one half until r exceeds $\frac{\sqrt{n}+\theta}{1-\lambda}$, where n is the number of components.

Each of the two models has its own intuition and logic that explain the cascading failure of the network. One can not easily determine which model is absolutely better in real world cases. So the good approximation between the two models lessens the burden of choosing one model.

The thesis also does more detailed research regarding the estimator of offspring distribution in the Poisson branching process model. A new and improved unbiased estimator can be used to get the better estimates of the cascade propagation parameter λ when applying the branching process model to real data.

The improved estimator for λ can be used in several ways. By using the estimate of λ to get the probability distributions of blackout sizes, the evaluation of risk in a given system model becomes possible. The system designer will be able to anticipate the risk of the system from past data and the insurance could be made on the basis of these results, once the models become further established.

The remaining problem of estimation is the bounds on the variance of estimate of λ.

Though the asymptotic convergence of variance is proved, the actual variance in the case of a practical number of components is not perfectly optimized. The current analysis is good for some limited cases but the other cases are left for the future work.

The properties that are dealt in this thesis are a step beyond previous researches. They enable the engineer to better understand the cascading failure phenomena of the blackout and will help to establish several tools to analyze them. Also they can be used in other fields that encounters similar cascading phenomena. The thesis deepens the understanding of cascading blackout of power systems and enables more accurate and efficient analysis.

Bibliography

[1] B.A. Carreras, V.E. Lynch, I. Dobson, D.E. Newman. Critical points and transitions in an electric power transmission model for cascading failure blackouts. In *Chaos, vol. 12, no. 4*, pages 985–994, December 2002.

[2] Ian Dobson, Benjamin A. Carreras, Vickie E. Lynch, David E. Newman. Complex systems analysis of series of blackouts: Cascading failure, critical points, and self-organization. In *Chaos, vol. 17, 026103*, June 2007.

[3] Ian Dobson, Benjamin A. Carreras, David E. Newman. A branching process approximation to cascading load-dependent system failure. In *37th Hawaii International Conference on System Science*, Hawaii, January 2004.

[4] Kevin R. Wierzbicki. Statistical estimation of cascading blackout size and propagation with branching process, MS thesis, University of Wisconsin-Madison. pages 21–22, May 2006.

[5] N.M. Yanev. On the Statistics of Branching Processes. In *Theory of Probability and its Applications, vol. 20*, pages 612–620, March 1975.

[6] Ian Dobson, Benjamin A. Carreras, Vickie E. Lynch, Bertrand Nkei, David E. Newman. Estimating failure propagation in models of cascading blackouts. In *Probability in the Engineering and Informational Sciences, vol. 19*, pages 475–488, Oct 2005.

[7] Jie Chen, James S. Thorp, Ian Dobson. Cascading dynamics and mitigation assessment in power disturbances via a hidden failure model. In *Electrical power and Energy Systems 27*, pages 318–326, May 2005.

[8] Dusko P. Nedic, Ian Dobson, Daniel S. Kirschen, Benjamin A. Carreras, Vickie E. Lynch. Criticality in a cascading failure blackout model. In *Electrical power and Energy Systems 28*, pages 627–633, Mar 2006.

[9] R. C. Hardiman, M. T. Kumbale, Y. V. Makarov. An advanced tool for analyzing multiple cascading failures. In *Eighth International Conference on Probability Methods Applied to Power Systems*, pages 629–634, Sep 2004.

[10] Dmitry N. Kosterev, Carson W. Taylor, William A. Mittelstadt. Model validation for the August 10, 1996 WSCC System Outage. In *IEEE Transactions on Power Systems, Vol 14*, pages 967–979, August 1999.

[11] Qiming Chen, Chuanwen Jiang, Wenzheng Qiu, James D. McCalley. Probability Models for Estimating the Probabilities of Cascading Outages in High-Voltage Transmission Network. In *IEEE Transactions on Power Systems, Vol 21*, pages 1423–1431, Aug 2006.

[12] Union for the co-ordination of transmission of electricity. Final Report System Disturbance on 4 November 2006. In *http://www.ucte.org/publications/otherreports/*, Jan 2007.

[13] U.S.-Canada Power System Outage Task Force. Final Report on the August 14th blackout in the United States and Canada: Causes and Recommendations. United States Department of Energy and National Resources Canada, April 2004.

[14] Simon A. Levin. New Directions in the Mathematics of Infectious Disease. In *Mathematical approaches for emerging and reemerging infectious diseases : models, methods, and theory*, pages 1–5, May 2002.

[15] Gina Kolata. Mathematical Model Predicts AIDS Spread. In *Science, New Series, Vol. 235, No. 4795*, pages 1464–1465, March 1987.

[16] A.D. Barbour, Sergey Utev. Approximating the Reed-Frost epidemic process. In *Stochastic Process Applications 113*, pages 173–197, May 2004.

[17] Ian Dobson, Kevin R. Wierzbicki, Janghoon Kim, Hui Ren. Towards Quantifying Cascading Blackout Risk. In *IREP Symposium Bulk Power System Dynamics and Control-VII*, pages 19–24, August 2007.

[18] Ian Dobson, Benjamin A. Carreras, David E. Newman. Branching process models for the exponentially increasing portions of cascading failure blackouts. In *38th Hawaii International conference on System Sciences*, Hawaii, January 2005.

[19] Xiaofeng Weng, Yiquang Hong, Ancheng Xue, Shengwei Mei. Failure analysis on China power grid based on power law. In *Journal of Control Theory and Applications, Vol. 4*, pages 235–238, August 2006.

[20] Ian Dobson, Benjamin A. Carreras, David E. Newman. A loading-dependent model of probabilistic cascading failure. In *Probability in the Engineering and Informational Sciences, vol. 19*, pages 15–32, Jan 2005.

[21] W. Feller. Stirling's Formula. An Introduction to Probability Theory and Its Applications, Vol. 1. pages 50–53, New York:Wiley, 1968.

Summary Page

This document has 1 figures.

This document has 2 table.

There are 6 pages in the preamble.

There are 33 pages in the body of the paper.

There are 3 pages in the bibliography.