CONSTRAINT AT A SADDLE NODE BIFURCATION

by

SCOTT GREENE

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Abstract

Many power engineering systems have dynamic state variables which can encounter constraints or limits affecting system stability. Voltage collapse is an instability associated with the occurrence of a saddle node bifurcation in the equations which model the electric power system. We investigate the effect of constraints on voltage collapse by studying the effect of constraining state variables on systems of nonlinear differential equations which exhibit saddle node bifurcations. When a dynamic state variable is constrained it becomes a constant and the system stability margin can change. We quantify the change in stability margin resulting from the constraint of one or several state variables. More precisely, we approximate the new distance to bifurcation in parameter space for a system which encounters a limit at a saddle node bifurcation. The emphasis of this thesis is on the derivation and interpretation of this calculation. Future work will test the applications of this calculation to various power system limits, the placement of voltage support, and model reduction for the study of voltage collapse.

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Introduction

Many power engineering systems have dynamic state variables which can encounter constraints or limits affecting system stability. Voltage collapse is an instability in heavily loaded power systems characterized by system wide catastrophically decreasing voltages. Voltage collapse is associated with the singularity of the equations which model the power system, and the occurrence of a saddle node bifurcation [5][23][8][9]. We investigate the effect of constraints on voltage collapse by studying the effect of constraining state variables on systems of nonlinear differential equations which exhibit saddle node bifurcations. Since a great variety of power system models exist, along with a myriad of possible constraints, we adopt a general mathematical hypothesis so that the analysis is not specific to any particular model or system. We concentrate on mathematical derivation and interpretation in order to provide a secure foundation for future work applying and testing the results on realistic systems.

Several examples of voltage collapse are described in [24][19][8][9][25], along with compilations of voltage collapse models and overviews of current industry practice. Incidents of voltage collapse usually correspond to a failure of generation and transmission to meet reactive power demand. As loads increase, a stable equilibrium solution of the modeling equations vanishes. The increase in load that causes a stable operating point to disappear in a saddle node bifurcation is referred to as the load power margin, or stability margin, at that point.

As system conditions change due to increasing loads, encountering limits can cause discontinuous change in stability margin without change in the system state. In other words, the system voltages and loading can be the same immediately before and after the constraint, yet the degree of stability of the system will change. Physical constraints, such as generator reactive power limits[6], and constraints resulting from operator control action, such as locking transformer tap changers[28][17][31][18], and the operation of HVDC links and Static VAR compensators[24], have been implicated in either the prevention or cause of voltage collapse. The effect of each possible constraint can be analyzed on a case by case basis, with separate numerical simulation for each possible contingency. This approach, however, is computationally expensive, since each constraint corresponds to a new set of model equations to be solved. In addition, exhaustive numerical computation does not provide any conceptual insight applicable from one system or constraint, to another.

The system stability margin may be measured by the change in parameters which produces bifurcation, or, equivalently, as the distance to bifurcation in parameter space. Consider a system which is about to become unstable due to a saddle node bifurcation. We are interested in quantifying the change in stability margin when one or several of the dynamic variables are constrained to constant values. More precisely, we approximate the new distance to bifurcation in parameter space for a system which encounters a limit at a saddle node bifurcation point.

The process of simplifying a system of equations is often analogous to the problem of modeling a physical system encountering a limit. That is, some useful model simplifications can be formulated as constraining variables to be constant in a system of equations. However, in simplifying a system of equations in this way, the variables to be constrained should be those for which the constraining has little effect on the system properties of interest. In our case, we are interested in simplifications which have little effect on the distance to bifurcation. Quantifying the effect of constraining each variable of the system enables the variables with little effect to be selected and constrained to constants, reducing the complexity of the model. For power system models, these simplifications not only make the equations more tractable but also suggest that the dynamics of the constrained variables are not significant factors in voltage collapse. Simplifying the model results in concise understanding of the phenomena being modeled.

In [3], the problem of a general dynamic system encountering a constraint is discussed and an expression indicating the stability of the system after encountering the constraint is presented. [18][6] address both generator limits and locking tap changers in electric power systems, and relates the problem of constraints to the generic theory of transcritical and saddle node bifurcations. In this thesis we approximate the change in stability margin of a general dynamic system encountering constraint of a state variable at a saddle node bifurcation by exploiting the bifurcation geometry of the system.

We first present assumptions and explain a mathematical scheme for incorporating constraints into the differential equations governing a system. We then derive an expression quantifying the effects of constraints in terms of the change in stability margin. Finally, alternate interpretations and derivations of the results are discussed, as well as connections to previous work, possible applications, and future work. The appendix contains the details of the proofs, illustrative examples, and an example of model reduction for a small power system.

1. Model and Assumptions

In this chapter we specify the mathematical tools and assumptions used to model the application of constraints to power systems. We consider the system described by the differential equation:

$$\dot{z} = F(z, \lambda), \quad z \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}^m \tag{1.1}$$

We assume F sufficiently smooth. For typical power system applications z is a state vector which includes bus voltage magnitudes and angles while λ is a parameter vector of real and reactive load powers. The parameter vector changes slowly compared to the dynamics of the system so that the state tracks a locus of stable equilibria. The set of (z, λ) for which

$$0 = F(z, \lambda) \tag{1.2}$$

define equilibria of the system, and the subset for which F_z has a distinct simple zero eigenvalue identify saddle node bifurcations. We assume that at (z_*, λ_*), F_z has a distinct simple zero eigenvalue and that (1.2) exhibits a generic saddle node bifurcation [12] satisfying the transversality conditions:

$$\mathbf{w}F_{zz}(\mathbf{v}, \mathbf{v}) \neq 0 \tag{1.3}$$

$$\mathbf{w}F_{\lambda} \neq 0 \tag{1.4}$$

where $\mathbf{v}$ and $\mathbf{w}$ represent the respective right and left eigenvectors corresponding to the zero eigenvalue of F_z. We adopt the normalization $\mathbf{wv} = 1$ with the sign convention that $\mathbf{w}F_{zz}(\mathbf{v}, \mathbf{v})$ is positive. We consider only the case $m = 1$, which is equivalent to regarding λ as a one dimensional parameter vector representative of a loading factor in a forecasted or critical direction of load increase. In addition, we assume $\lambda_* = 0$ and that the system tracks a stable equilibrium for negative λ. The map F encompasses both static and dynamic conditions for equilibria. That is to say, for power system applications, solutions of (1.2) for which F_z has all eigenvalues with negative real parts satisfy both the load flow equations and stable equilibrium conditions of the system dynamics. However, since bifurcation of the load flow equations implies bifurcation of the dynamic model[4], it is often sufficient in this thesis to consider F as simply reflecting static equations (see Appendix 10). The relationship between voltage collapse, the saddle node bifurcation, and the dynamic and static modeling equations is discussed in [4] and [5].

We wish to model the application of a constraint on a single state variable, or combination of state variables. Let $x \in \mathbb{R}^s$, $y \in \mathbb{R}^{n-s}$ represent a partition of the state vector z , so that $z = (x, y)$. We assume that the constraint limits the state variables x to their bifurcation values, x_*, when the system is at the bifurcation point (x_*, y_*, λ_*). The associated partition of (1.2) is:

$$\begin{aligned} 0 &= f(x, y, \lambda) \\ 0 &= g(x, y, \lambda) \end{aligned} \tag{1.5}$$

with the map $F(z, \lambda)$ considered as two separate maps $f : \mathbb{R}^s \times \mathbb{R}^{n-s} \times \mathbb{R} \to \mathbb{R}^s$ and $g : \mathbb{R}^s \times \mathbb{R}^{n-s} \times \mathbb{R} \to \mathbb{R}^{n-s}$

We assume that f_x is non-singular at the bifurcation,

$$\left| f_x \right| \neq 0 \tag{1.6}$$

and we write the Jacobian matrix as:

$$F_z = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \tag{1.7}$$

This partitioning of the Jacobian matrix, for f_x invertible, allows for a convenient expression for its determinant:

$$\left| F_z \right| = \left| f_x \right| \left| g_y - g_x [f_x]^{-1} f_y \right| \tag{1.8}$$

The matrix $g_y - g_x[f_x]^{-1} f_y$ is called the Schur complement of the matrix f_x, and this expression for the determinant results from Schur's Theorem [14]. Note that F_z is singular if and only if the Schur complement of f_x is singular. In Appendix 1 we show that the y-partitions of the zero eigenvectors of F_z are zero eigenvectors of the Schur complement of f_x, and that the Schur complement of f_x has a distinct simple zero eigenvalue when its left and right zero eigenvectors are not orthogonal. We assume that the y–partitions of the left and right zero eigenvectors of F_z are not orthogonal to ensure that the Schur complement of f_x has a unique simple zero eigenvalue.

After encountering the constraint, the x components no longer appear as state variables. The constrained system equilibria are the solutions (y, λ) of

$$g(x_*, y, \lambda) = 0 \tag{1.9}$$

and an operating point of the constrained system is stable if all the eigenvalues of $g_y|_{(y_*, \lambda_*, \theta_*)}$ have negative real parts. A saddle node bifurcation point of the constrained system, (y_{c*}, λ_{c*}), is a solution of (1.9) for which the matrix g_y evaluated at $(x_*, y_{c*}, \lambda_{c*})$ has one simple zero eigenvalue.

We proceed to the details of modeling the constraint of x. Since f_x is invertible in a neighborhood of (z_*, λ_*), by the implicit function theorem[22] there is a neighborhood $S \times P$ of (y_*, λ_*) and a function $h : S \times P \to \mathbb{R}^s$ such that $f\big(h(y,\lambda), y, \lambda\big) = 0$. The implicit function $h(y, \lambda)$ is smooth with a derivative given by,

$$h_y = -[f_x]^{-1} f_y \tag{1.10}$$

Thus, in a neighborhood of (z_*, λ_*), solutions of (1.2) satisfy:

$$0 = g(h(y, \lambda), y, \lambda) \tag{1.11}$$

and solutions of (1.11) can be uniquely identified with solutions of (1.2) using the implicit function,

$$x = h(y, \lambda) \tag{1.12}$$

Note in particular that (1.8) implies that the bifurcation point (z_*, λ_*) of (1.2) corresponds to the bifurcation point (y_*, λ_*) of (1.11). We approximate the application of a constraint by construction of a homotopy joining the unconstrained system defined by (1.11) to the constrained system defined by (1.9). Consider the map:

$$\Theta : \mathbb{R}^s \times [0, 1] \to \mathbb{R}^s$$

$$\Theta(x, \theta) = x(1 - \theta) + \theta x_* \tag{1.13}$$

The abstract parameter θ can be thought of as a measure of the extent to which x is constrained to x_*. We define a new map,

$$H : S \times P \times [0, 1] \to \mathbb{R}^{n-s}$$

$$H(y, \lambda, \theta) = g\big(\Theta(h(y, \lambda), \theta), y, \lambda\big) \tag{1.14}$$

with linearization,

$$H_y = g_y - (1 - \theta) g_x [f_x]^{-1} f_y \tag{1.15}$$

Equilibria of the constrained and unconstrained systems are solutions of:

$$0 = H(y, \lambda, \theta) \tag{1.16}$$

Solutions of (1.16) for $\theta = 0$ correspond to solutions of (1.11), which can be put into one–to–one correspondence with the solutions of (1.2); similarly, solutions for

$\theta = 1$ correspond to solutions of (1.9). Notice that for the case $\theta = 1$, $H_y = g_y$ and that for $\theta = 0$, $H_y = g_y - g_x[f_x]^{-1}f_x$, the Schur complement of f_x for the original Jacobian matrix. Thus, bifurcations of (1.16) for $\theta = 0$ correspond to bifurcations of (1.2), and bifurcations of (1.16) for $\theta = 1$ correspond to bifurcations of (1.9). We write $\theta_* = 0$ and $\theta_{c*} = 1$, to identify the value of θ for the unconstrained and constrained system bifurcations respectively. In Appendix 3 we show that the saddle node bifurcation of (1.16) at $(y_*, \lambda_*, \theta_*)$ satisfies the transversality conditions of the generic saddle node bifurcation

$$wH_{yy}(v, v) \neq 0 \tag{1.17}$$

$$wH_\lambda \neq 0 \tag{1.18}$$

where w and v are left and right zero eigenvectors respectively of H_y at $(y_*, \lambda_*, \theta_*)$.

Observe that the bifurcating equilibrium, (y_*, λ_*), of the unconstrained system must also be an equilibrium of the constrained system but in general is not a bifurcation point for the constrained system. Indeed, we would like to know how far (y_*, λ_*) is from the new bifurcation, (y_{c*}, λ_{c*}). Our goal is to find a calculation based on information available from the unconstrained system (1.2), that allows us to estimate the increase in parameter distance to bifurcation, λ_{c*}, upon encountering the constraint.

2. Derivation

In this chapter we derive an approximation for the change in parameter margin to saddle node bifurcation for the system with equilibrium described by (1.2),

$$0 = F(z, \lambda) \tag{1.2}$$

upon encountering the constraint on the x variables.

The set of equilibrium points of (1.16),

$$0 = H(y, \lambda, \theta) \tag{1.16}$$

for which H_y is also singular with a unique simple zero eigenvalue identifies the set of saddle node bifurcation points of (1.16). Since F_z has a unique simple zero eigenvalue at $(z_*, \lambda_*, \theta_*)$, H_y also has at least one zero eigenvalue at $(y_*, \lambda_*, \theta_*)$. We show in Appendix 1 that typically H_y will also have a unique simple zero eigenvalue, and we restrict our analysis to this case. Thus, for some neighborhood $N \subset S \times P \times [0,1]$ about $(y_*, \lambda_*, \theta_*)$, H_y has a real eigenvalue, $\mu : N \to \mathbb{R}$, that is a smooth function of (y, λ, θ) such that $\mu(y_*, \lambda_*, \theta_*) = 0$. Then,

$$\mu(y, \lambda, \theta) = w(y, \lambda, \theta) H_y(y, \lambda, \theta) v(y, \lambda, \theta) \tag{2.1}$$

where w and v represent the left and right eigenvectors of H_y corresponding to μ, normalized so that $wv = 1$. For normalized zero eigenvectors of H_y at $(y_*, \lambda_*, \theta_*)$ we use

$$v = \mathbf{v}^y \tag{2.2}$$

$$w = \frac{\mathbf{w}^y}{\mathbf{w}^y \mathbf{v}^y} \tag{2.3}$$

where $\mathbf{w}^y$ and $\mathbf{v}^y$ are the y–partitions of $\mathbf{w}$ and $\mathbf{v}$ respectively. Note that the assumption that H_y has a unique simple zero eigenvalue implies that $\mathbf{w}^y \mathbf{v}^y \neq 0$. Although we assume that the critical eigenvalue of F_z is negative for negative λ, that is not necessarily the case for μ. The product $\mathbf{w}^y \mathbf{v}^y$ indicates from which direction the critical eigenvalue μ of H_y approaches zero with increasing λ (see Appendix 3). For $\mathbf{w}^y \mathbf{v}^y$ positive, μ is negative for negative λ.

The set of bifurcation points of (1.16) in the neighborhood N are solutions of (1.16) for which μ evaluates to zero. We denote this set of bifurcation points of (1.16) by Σ,

$$\Sigma = \left\{ (y, \lambda, \theta) \;\middle|\; (y, \lambda, \theta) \in H^{-1}(0) \cap \mu^{-1}(0) \right\} \tag{2.4}$$

Consider the map

$$U : N \to \mathbb{R}^{n-s} \times \mathbb{R}$$
$$U(y, \lambda, \theta) = \begin{pmatrix} H(y, \lambda, \theta) \\ \mu(y, \lambda, \theta) \end{pmatrix} \tag{2.5}$$

Then we may identify Σ with the zero section of U

$$\Sigma = U^{-1}(0) \tag{2.6}$$

By the implicit function theorem, since the matrix (U_y, U_λ) is invertible at $(y_*, \lambda_*, \theta_*)$ (see Appendix 4), there is a neighborhood T of θ_* and smooth functions

$$Y : T \to \mathbb{R}^{n-s} \qquad \Lambda : T \to \mathbb{R} \tag{2.7}$$

such that $\big\{(Y(\theta), \Lambda(\theta), \theta) \bigm| \theta \in T\big\} \subset \Sigma$ and

$$U\big(Y(\theta), \Lambda(\theta), \theta\big) = 0 \tag{2.8}$$

Thus, the bifurcation set Σ is a curve in a neighborhood of $(y_*, \lambda_*, \theta_*)$ that we can parameterize with the maps $Y(\theta)$ and $\Lambda(\theta)$ implicitly defined by

$$H(Y(\theta), \Lambda(\theta), \theta) = 0 \tag{2.9}$$
$$w(Y(\theta), \Lambda(\theta), \theta) H_y(Y(\theta), \Lambda(\theta), \theta) v(Y(\theta), \Lambda(\theta), \theta) = 0 \tag{2.10}$$

The map $\Lambda(\theta)$ describes the relation of λ to θ on the bifurcation set in a neighborhood of $(y_*, \lambda_*, \theta_*)$. We approximate $\Lambda(\theta)$ by a Taylor series expansion about $(y_*, \lambda_*, \theta_*)$, which, including only to third order terms and noting that $\Lambda(0) = 0$, is

$$\Lambda = \Lambda_\theta \theta + \frac{1}{2}\Lambda_{\theta\theta}\theta^2 + \frac{1}{6}\Lambda_{\theta\theta\theta}\theta^3 \tag{2.11}$$

Differentiating (2.9) at $(y_*, \lambda_*, \theta_*)$ gives,

$$H_y Y_\theta + H_\lambda \Lambda_\theta + H_\theta = 0 \tag{2.12}$$

where for readability, we have dropped notation indicating parameterization by θ. Differentiation of (2.12) yields,

$$\begin{aligned} 0 =& H_{yy} Y_\theta Y_\theta + 2H_{y\lambda} Y_\theta \Lambda_\theta + 2H_{y\theta} Y_\theta + H_y Y_{\theta\theta} \\ &+ H_{\lambda\lambda}\Lambda_\theta\Lambda_\theta + 2H_{\lambda\theta}\Lambda_\theta + H_\lambda \Lambda_{\theta\theta} + H_{\theta\theta} \end{aligned} \tag{2.13}$$

We now evaluate these equations at $(y_*, \lambda_*, \theta_*)$, and solve for the derivatives of Λ. Differentiating (2.10) at $(y_*, \lambda_*, \theta_*)$ and omitting terms that vanish since they contain wH_y or $H_y v$ gives,

$$wH_{yy}Y_\theta v + wH_{y\lambda}\Lambda_\theta v + wH_{y\theta}v = 0 \tag{2.14}$$

The derivatives of H involve the derivatives and products of the functions that are used to define H, the complete derivation of which is given in the appendix. Since $H_\theta = 0$ at $(y_*, \lambda_*, \theta_*)$ (see Appendix 2), premultiplying (2.12) by w leads to

$$wH_\lambda \Lambda_\theta = 0, \tag{2.15}$$

which with (1.4) implies $\Lambda_\theta = 0$. For $\Lambda_\theta = 0$, (2.12) becomes

$$H_y Y_\theta = 0, \tag{2.16}$$

implying that Y_θ is collinear to v at $(y_*, \lambda_*, \theta_*)$. We write $Y_\theta = \alpha v$, for some scalar α. Thus, (2.14) becomes,

$$wH_{yy}(v, v)\alpha + wH_{y\theta}v = 0 \tag{2.17}$$

Solving for α then gives

$$\alpha = -\frac{wH_{y\theta}v}{wH_{yy}(v, v)} \tag{2.18}$$

Premultiplying (2.13) by w and eliminating terms that evaluate to zero at $(y_*, \lambda_*, \theta_*)$ leaves,

$$wH_{yy}Y_\theta Y_\theta + 2wH_{y\theta}Y_\theta + wH_\lambda \Lambda_{\theta\theta} = 0 \tag{2.19}$$

Substituting $Y_\theta = \alpha v$ into (2.19) and solving for $\Lambda_{\theta\theta}$ yields

$$\Lambda_{\theta\theta} = \frac{(wH_{y\theta}v)^2}{wH_{yy}(v, v)wH_\lambda} \tag{2.20}$$

which can be expressed (see Appendix 3) in terms from the original system(1.1) evaluated at the bifurcation,

$$\Lambda_{\theta\theta} = \frac{(\mathbf{w}^x f_x \mathbf{v}^x)^2}{\mathbf{w}F_{zz}(\mathbf{v}, \mathbf{v})\ \mathbf{w}F_\lambda} \tag{2.21}$$

Substitution of (2.20) into (2.13) leads to an expression involving $Y_{\theta\theta}$ which, with differentiation of (2.13) allows for computation of $\Lambda_{\theta\theta\theta}$ (see Appendix 5),

$$\begin{aligned}\Lambda_{\theta\theta\theta} =&(wH_{y\theta}v)^3\left[\frac{wH_{yyy}(v,v,v)}{(wH_{yy}(v,v))^3 wH_\lambda}+\frac{wH_{y\lambda}v}{(wH_{yy}(v,v)wH_\lambda)^2}\right]\\ &-(wH_{y\theta}v)^2\left[\frac{wH_{yy\theta}(v,v)+wH_{\lambda\theta}}{(wH_{yy}(v,v))^3 wH_\lambda}\right]\\ &+3(wH_{y\theta}v)\left[\frac{wH_{yy}v}{wH_{yy}(v,v)}-\frac{wH_{y\theta}}{wH_{y\theta}v}\right]Y_{\theta\theta}\end{aligned}\tag{2.22}$$

Since $\Lambda_\theta = 0$ at $(y_*, \lambda_*, \theta_*)$, a third order Taylor series approximation for Λ about $(y_*, \lambda_*, \theta_*)$ is:

$$\Lambda(\theta) = \frac{1}{2}\Lambda_{\theta\theta}\theta^2 + \frac{1}{6}\Lambda_{\theta\theta\theta}\theta^3 \tag{2.23}$$

We approximate the change in parameter distance to bifurcation, λ_{c*}, by extrapolating (2.23) to $\theta = 1$. Evaluation of $\Lambda_{\theta\theta\theta}$ requires numerically computing the components of $Y_{\theta\theta}$ not in the v direction (see Appendix 5). However, a second order approximation and (2.21) leads to an easily computable closed form estimate for the new security margin:

$$\lambda_{c*} = \frac{(\mathbf{w}^x f_x \mathbf{v}^x)^2}{2\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})\ \mathbf{w}F_\lambda} \tag{2.24}$$

The quantity $wH_{y\theta}v$ in the numerator of (2.20) is the sensitivity to θ of the zero eigenvalue of H_y at the bifurcation $(y_*, \lambda_*, \theta_*)$. Differentiation of (2.1) with respect to θ at $(y_*, \lambda_*, \theta_*)$ and noting that $wH_y = 0$ and $H_y v = 0$ at $(y_*, \lambda_*, \theta_*)$ yields

$$\mu_\theta = wH_{y\theta}v \tag{2.25}$$

which can be expressed in terms of the original system (see Appendix 3):

$$\mu_\theta = \frac{\mathbf{w}^x f_x \mathbf{v}^x}{\mathbf{w}^y \mathbf{v}^y} \tag{2.26}$$

The sign of (2.25) indicates in which direction to first order the critical eigenvalue of H_y moves as θ increases from zero. Thus, since the sign of $\mathbf{w}^y\mathbf{v}^y$ describes the relation between the directions in which the critical eigenvalues of H_y and F_z move as the bifurcation occurs, the sign of $\mathbf{w}^x f_x \mathbf{v}^x$ indicates in which direction the greatest real eigenvalue of F_z moves as the constraint is applied. Thus, for a constraint to be stabilizing, $\mathbf{w}^x f_x \mathbf{v}^x$ must be negative. However, negative $\mathbf{w}^x f_x \mathbf{v}^x$ only implies that the eigenvalue closest to the origin of the unconstrained system is negative for the constrained system, not that all the constrained system eigenvalues are negative. The constrained system is stable only if all the eigenvalues of $g_y|_{(y_*,\lambda_*,\theta_*)}$ are negative.

3. Discussion

In the previous two chapters we constructed a method for the analysis of constraints on power systems. The method used the implicit function theorem to express the variables to be constrained in terms of the unconstrained variables and load parameter, and then related the constrained variables to their limit values by using a smooth homotopy to represent the application of the constraint. Then we derived expressions relating the unconstrained variables to the load and constraint parameters on the bifurcation set. From these expressions we deduced the coefficients of a Taylor series representation for the map relating the load parameter to the constraint parameter in a neighborhood of the original bifurcating equilibrium. The main result (2.24) describes to second order the parameter margin to bifurcation of the constrained system

$$\lambda_{c*} = \frac{(\mathbf{w}^x f_x \mathbf{v}^x)^2}{2\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})\ \mathbf{w}F_\lambda} \tag{2.24}$$

where the constraint is stabilizing if the sign of $\mathbf{w}^x f_x \mathbf{v}^x$ is negative. In addition we showed that (2.24) can be refined to include third order terms. However, (2.24) is not only computationally simple, but contains terms with geometric interpretations that illustrate alternate interpretations and derivations. We note two informal, but informative derivations.

Consider the generic saddle node bifurcation diagram (Figure 1) of the original system (1.2). The variable s represents the Lyapunov–Schmidt reduced state variable, a scalar representative of the projection of the equilibrium position onto the right zero eigenvector at the bifurcation. Explanation of the Lyapunov–Schmidt reduction or reduction to the center manifold can be found in [29][2][12], and a brief elementary description is given in [18] and Appendix 6. The bifurcation diagram is the graph of s vs. λ for

$$L(s,\lambda) = 0 \tag{3.1}$$

where L represents the reduction of F to the center manifold.

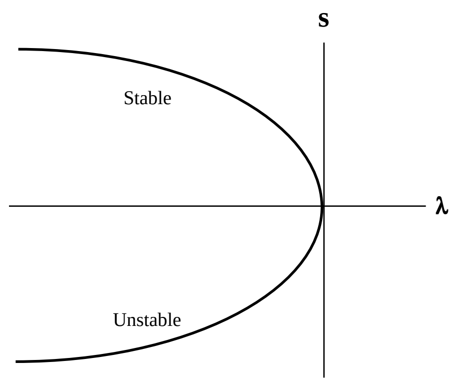

Figure 1. Saddle node bifurcation diagram

The second order Taylor series for (3.1) about the bifurcation is

$$\frac{1}{2}L_{ss}s^2 + \frac{1}{2}L_{\lambda\lambda}\lambda^2 + L_{s\lambda}s\lambda + L_\lambda\lambda = 0 \tag{3.2}$$

where the derivatives of L are evaluated at the bifurcation $(s_*, \lambda_*) = (0, 0)$. We can approximate the bifurcation diagram near the bifurcation (see Appendix 6) as

$$0 = \frac{1}{2}L_{ss}s^2 + L_\lambda\lambda \tag{3.3}$$

or

$$\lambda = -\frac{L_{ss}s^2}{2L_\lambda} \tag{3.4}$$

where

$$L_{ss} = \mathbf{w}F_{zz}(\mathbf{v}, \mathbf{v}), \qquad L_\lambda = \mathbf{w}F_\lambda \tag{3.5}$$

If we parameterize the solutions of (3.1) by s we note that the curvature, κ, of the bifurcation diagram at the bifurcation depends on L_{ss} (see Appendix 6)

$$\kappa = -L_{ss} = -\mathbf{w}F_{zz}(\mathbf{v}, \mathbf{v}) \tag{3.6}$$

where the minus sign results from our selection of a sign convention for curvature and the assumption that solutions exist only for negative λ. Observe that (3.4) and the hypothesis that no solutions exist for positive λ implies that $\mathbf{w}F_\lambda$ has the same sign as $\mathbf{w}F_{zz}(\mathbf{v}, \mathbf{v})$. The gradient of L near the bifurcation is obtained from (3.3),

$$DL = (L_{ss}s, L_\lambda) \tag{3.7}$$

and at the bifurcation, the gradient is precisely

$$DL|_{s=0,\lambda=0} = (0, L_\lambda) = (0, \mathbf{w}F_\lambda) \tag{3.8}$$

The latter observation may be generalized to the multidimensional load parameter case to show that $\mathbf{w}F_\lambda$ represents a normal vector to the bifurcation set [4], although for one dimensional λ, this interpretation is less useful. The effect of changing $\mathbf{w}F_\lambda$ on the bifurcation diagram can be visualized by observing that as $\mathbf{w}F_\lambda$ increases, the λ component of the normal vector to the bifurcation diagram increases relative to the s component and hence the nose of the s vs. λ graph flattens. Similarly, as $\mathbf{w}F_\lambda$ decreases, the nose becomes more pronounced. Thus, $\mathbf{w}F_\lambda$ can be thought of as an indicator of how rapidly the bifurcation appears with increasing load. Note that as $\mathbf{w}F_\lambda$ increases, λ_{c*} decreases, and $\mathbf{w}F_\lambda$ is greatest when the vector F_λ is parallel to $\mathbf{w}$.

The bifurcation diagram provides a simple picture illustrating the effect of a constraint on the margin to bifurcation (Figure 2). Under the hypothesis of Chapter 1, both the constrained and unconstrained systems exhibit saddle node bifurcations, so both systems can be reduced to their center manifolds. Thus, both systems have a corresponding graph similar to Figure 1. Since the bifurcation of the unconstrained system is also an equilibrium of the constrained system, the two graphs intersect at the nose of the unconstrained bifurcation diagram. The change in stability margin is the distance from the nose of the unconstrained system bifurcation diagram to the nose of the constrained system bifurcation diagram. Figure 2 provides a visual aid for the following two derivations of (2.24).

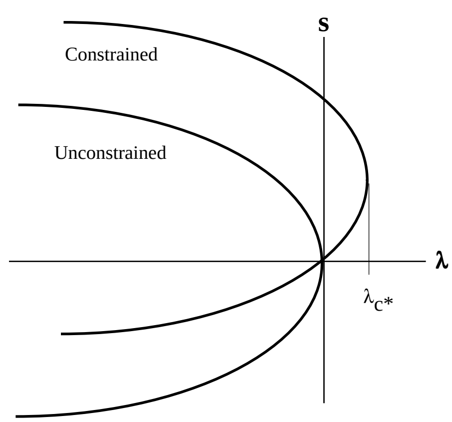

Figure 2. Bifurcation diagrams of constrained and unconstrained systems

The eigenvalue of the Lyapunov–Schmidt reduced system is the same as the greatest real eigenvalue of the complete system Jacobian matrix, F_z. Thus, the variable s can be directly related to μ, the greatest real eigenvalue of F_z, which we approximate by a Taylor series about the bifurcation computed from differentiation of (3.3),

$$\mu = D_s L = L_{ss} s \qquad s = \frac{\mu}{L_{ss}} \tag{3.9}$$

Substitution into (3.4) approximates the relation of μ to λ near the bifurcation

$$\lambda = -\frac{\mu^2}{2L_{ss}L_\lambda} = \frac{-\mu^2}{2\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})\ \mathbf{w}F_\lambda} \tag{3.10}$$

Given the effect of a constraint on the greatest real eigenvalue, we can use (3.10) to approximate the change in load power margin,

$$\Delta\lambda = \frac{(\Delta\mu)^2}{2\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})\ \mathbf{w}F_\lambda} \tag{3.11}$$

We now propose a different way to represent the constraint than presented previously. Since the constraint eliminates the dynamics of the constrained variables, we represent the constraint by eliminating the equations describing these dynamics from (1.1). The effect of the constraint on the system Jacobian matrix F_z then is to zero the rows and columns corresponding to the constrained variables. Thus,

$$\Delta f_x = -f_x \quad \Delta f_y = -f_y \quad \Delta g_x = -g_x \tag{3.12}$$

We then approximate $\Delta\mu$ by

$$\Delta\mu = \frac{\partial\mu}{\partial f_x}\Delta f_x + \frac{\partial\mu}{\partial g_x}\Delta g_x + \Delta f_y \frac{\partial\mu}{\partial f_y} \tag{3.13}$$

where $\frac{\partial\mu}{\partial f_x}$, $\frac{\partial\mu}{\partial g_x}$, and $\frac{\partial\mu}{\partial f_y}$ are the sensitivities of μ to change in the respective partitions of the Jacobian matrix. These sensitivities are the products of the elements of the corresponding left and right eigenvectors of F_z [14], and have also been called participation factors [27][21][16][10][15] or residues[13]. To simplify notation, we restrict to the case of a constraint on only one variable, although a constraint on multiple variables leads to the same result. Thus, $\frac{\partial\mu}{\partial f_x}$ is a scalar, $\frac{\partial\mu}{\partial f_y}$ is a column vector, and $\frac{\partial\mu}{\partial g_x}$ is a row vector

$$\frac{\partial\mu}{\partial f_x} = \mathbf{w}^x\mathbf{v}^x \quad \frac{\partial\mu}{\partial f_y} = \mathbf{w}^x\mathbf{v}^y \quad \frac{\partial\mu}{\partial g_x} = \mathbf{w}^y\mathbf{v}^x$$

At the bifurcation

$$\mathbf{w}^x f_x = -\mathbf{w}^y g_x \quad f_x \mathbf{v}^x = -f_y \mathbf{v}^y$$

thus, (3.13) reduces to

$$\Delta\mu = \mathbf{w}^x f_x \mathbf{v}^x \tag{3.14}$$

The sign of $\Delta\mu$ indicates in which direction the zero eigenvalue moves upon application of the constraint. Substitution of (3.14) into (3.11) gives the same result as derived in the previous chapter

$$\lambda_{c*} = \frac{(\mathbf{w}^x f_x \mathbf{v}^x)^2}{2\mathbf{w}F_{zz}(\mathbf{v}, \mathbf{v})\ \mathbf{w}F_\lambda} \tag{2.24}$$

Although this derivation is not rigorous, it illustrates that the previous result agrees with consideration of the saddle node bifurcation geometry as related to the greatest real eigenvalue. Note that this derivation does not require f_x invertible at the bifurcation nor that the $y-$partitions of the zero eigenvectors of F_z not be orthogonal.

We now derive (2.24) from another perspective, using the observation that the homotopy relating the constrained and unconstrained systems creates a transcritical bifurcation[6][18]. While the saddle node bifurcation occurs when change in a parameter causes a stable equilibrium (the node) and an unstable equilibrium(the saddle) to combine and disappear, the transcritical bifurcation involves the exchange of stability rather than the annihilation of equilibria. For the transcritical bifurcation, an equilibrium remains fixed as the parameter varies, but the stability of that equilibrium changes as the parameter moves through the bifurcation. In particular, if (x_0, y_0, λ_0) is a solution of (1.2),

$$0 = F(z, \lambda) \tag{1.2}$$

then (y_0, λ_0, θ) is also a solution of (1.16) for any θ.

$$0 = H(y, \lambda, \theta) \tag{1.16}$$

Similarly, if (x_*, y_*, λ_*) is a saddle node bifurcation point of (1.2), then (y_*, λ_*, θ) is a saddle node bifurcation point of (1.16) for $\theta = 0$, and a solution of (1.16) for all θ. In Appendix 7 we show that (1.16) satisfies the transversality conditions for a transcritical bifurcation.

The transcritical bifurcation diagram (Figure 3) is the graph of s vs. θ for

$$L^t(s,\theta) = 0 \tag{3.15}$$

where L^t represents the reduction of H to the center manifold and s is again the projection of the equilibrium onto the right zero eigenvector at the bifurcation.

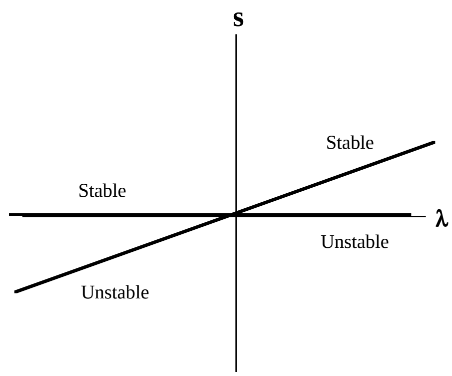

Figure 3. Transcritical bifurcation diagram

The Taylor series approximation for the Lyapunov–Schmidt reduction of the transcritical bifurcation is

$$\frac{1}{2}L^t_{ss}s^2 + L^t_{s\theta}\theta s = 0 \tag{3.16}$$

where L^t_{ss} and $L^t_{s\theta}$ evaluated at the bifurcation (see Appendix 7 and Appendix 3) are,

$$L^t_{ss} = wH_{yy}(v,v) = \frac{\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})}{\mathbf{w}^y\mathbf{v}^y}, \qquad L^t_{s\theta} = wH_{y\theta}v = \frac{\mathbf{w}^x f_x \mathbf{v}^x}{\mathbf{w}^y\mathbf{v}^y} \tag{3.17}$$

The distance between the two solutions on the $s-\theta$ transcritical bifurcation diagram is the distance between the solutions on the $s-\lambda$ saddle node bifurcation diagram, since s essentially represents projection onto the same subspace in both (3.1) and (3.15) (see Appendix 7). Thus, we solve (3.16) for s and substitute into (3.4) to find the distance to the saddle node bifurcation point. For $\theta = 0$, the constraint has not been applied, and (3.16) implies that $s = 0$ and hence, (3.4) gives $\lambda = 0$ as expected. For $\theta = 1$, we solve (3.16) to get $s_1 = 0$ and

$$s_2 = \frac{-2L^t_{s\theta}}{L^t_{ss}} = \frac{-2\mathbf{w}^x f_x \mathbf{v}^x}{\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})} \tag{3.18}$$

Thus the distance between the two solutions on the transcritical bifurcation diagram is

$$\Delta s = \frac{2\mathbf{w}^x f_x \mathbf{v}^x}{\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})} \tag{3.19}$$

corresponding to two roots of the saddle node bifurcation diagram at

$$s = \pm\frac{\mathbf{w}^x f_x \mathbf{v}^x}{\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})} \tag{3.20}$$

Substitution into (3.4) yields

$$\lambda_{c*} = \frac{(\mathbf{w}^x f_x \mathbf{v}^x)^2}{2\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})\ \mathbf{w}F_\lambda} \tag{2.24}$$

as previously derived. Note that the key assumption of this derivation is that the coefficients of s and λ in (3.3) and (3.4) do not change with θ. Also, since this derivation uses (1.16), it requires the same hypothesis and assumptions as the derivation presented in Chapter 2 and Chapter 3.

The previous derivations establish interpretations of the terms that appear in (2.24). The term $\mathbf{w}F_\lambda$ in the denominator of λ_{c*} in (2.24) is related to the rate at which the bifurcation progresses as loads increase in the given direction. The term $\mathbf{w}^x f_x \mathbf{v}^x$ in the numerator of λ_{c*} in (2.24) is related to the change in the greatest real eigenvalue of the linearization of the system. This quantity is also useful in that, while the calculation of the change in stability margin (2.24) is always positive, the sign of $\mathbf{w}^x f_x \mathbf{v}^x$ indicates the direction the zero eigenvalue moves with the constraint. Thus, if all other eigenvalues remain negative, the sign of $\mathbf{w}^x f_x \mathbf{v}^x$ determines the stability of the constrained system. The ratio of $\mathbf{w}^x f_x \mathbf{v}^x$ to $\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})$ is related to the distance between corresponding stable and unstable equilibria on both the saddle node and transcritical bifurcation diagrams. The term $\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})$ is also the curvature of the saddle node bifurcation diagram at the bifurcation point.

The denominator of λ_{c*} in (2.24), $2\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})\ \mathbf{w}F_\lambda$ remains the same for each possible constraint. Thus, if we wish to compare relative effects of different constraints, we need only consider the numerators. In addition, although we assumed a specific normalization and sign convention for our derivation, the evaluation of (2.24) is invariant to this selection. That is to say, for any left and right zero eigenvectors, (2.24) yields the same result for λ_{c*}. In fact, as a consequence of the geometric entities represented by the terms in (2.24), (2.24) is also a geometric invariant; evaluation of λ_{c*} from (2.24) does not change with transformation of coordinates.

4. Conclusion

In the preceding chapters we have defined a model for constraints on a dynamic system, derived an approximation for the change in stability margin for a system encountering a limit of a state variable at a saddle node bifurcation, and described alternate derivations and interpretations of the result. The theory is intended to provide a mathematical foundation for the analysis of constraints as they affect voltage collapse in electric power systems. In conclusion we outline possible applications for our theory and note considerations that require further attention.

We discuss one promising potential application. Consider the determination of the optimum bus to place reactive power support intended to increase stability margin. Typically, a solution can be approached by repeatedly solving a steady state stability simulation for the point of collapse [1] for each system with the compensation at a different bus. If we assume that the effect of reactive power support at a bus can roughly be modeled as the constraint of the voltage at that bus, then computation of (2.24) for each bus allows for an estimate in the security margin gained by providing compensation at that bus. [21][10][20][11] use participation factors, eigensensitivity, and modal analysis to determine candidate buses for support. We propose the use of (2.24) and briefly mention the potential advantages offered by (2.24). First, since computation of participation factors requires computing the right and left zero eigenvectors at the bifurcation, computation of (2.24) does not require any significant extra computation other than the evaluation of F_{zz} at the bifurcation. Note that F_{zz} is constant for the load flow equations in rectangular coordinates. In addition, (2.24) not only gives relative effects of constraining different bus voltages, but an estimate of the security margin gained, allowing for economic analysis of the benefit of providing support. In the case of power system models that include both algebraic and differential equations, (2.24) is invariant to scaling of the algebraic equations and hence provides consistent answers while participation factors do not (see Appendix 9). One of the present limitations in applying (2.24) to this problem is that (2.24) assumes that the bus voltage would be constrained to its value at the bifurcation.

As noted earlier, another possible application of this theory is to model reduction. A model for a particular power system can be obtained by combining detailed models of all the component systems that make up the power system. For example, a typical power system model would include swing dynamics for each generator and power balance at each bus; however, more comprehensive models can be obtained by including dynamics for loads, excitation systems, tap changing transformers, and control schemes. The system models that need to be included in the comprehensive model depend upon the physical phenomena to be simulated. If the purpose of the simulation is to study the steady state stability of the power system, then computation of (2.24) for each state variable of the comprehensive model identifies which variables could be removed with little effect on the stability margin, and thus do not participate in voltage collapse. An example of model reduction for a small power system can be found in Appendix 8 where it is demonstrated that the swing dynamics of the generator have little influence on the stability margin of the small power system.

Several theoretical limitations must be addressed to permit the successful application of the theory. First, the derivations assumed that the system encounters a constraint at a saddle node bifurcation. However, in power system applications it is likely that limits will be encountered before the system actually reaches voltage collapse. Extending the theory to include constraints away from bifurcation would be an important practical advantage and should be a priority for future work. In addition, it is necessary to identify which power system constraints the model used in this thesis applies to, and which it does not. The theory assumes that a constraint holds a state variable fixed at a particular value. Some constraints might be better modeled as replacing one dynamic equation with another, not eliminating dynamics. Constraints that can not be modeled as state variables becoming fixed at limiting values should be identified and analogous results derived.

The main result of this thesis, equation (2.24), is an approximation of the change in security margin when the system reaches a constraint. Thus, any application of the theory must be concerned with the accuracy of this approximation. We describe possible causes of inaccuracy. The derivations assumed that the particular eigenvalue or mode that caused bifurcation of the system was the same mode that would cause bifurcation of the constrained system. However, it may be possible that the eigenvalue of the unconstrained system that becomes zero at the bifurcation, is not the eigenvalue that becomes zero for the constrained system. The validity of this assumption is dependent upon the proximity of other bifurcations in $y-\lambda$ space to the original bifurcation, and the requirement that the zero eigenvalue of the Schur's complement of f_x in F_z was simple and unique (see Appendix 1). One should not expect the result to be accurate if these assertions are not true. The approximation may also prove inaccurate if the bifurcation geometry of the constrained system is much different than that of the unconstrained system. The derivation presented in Chapter 2 was also arrived at independently in Chapter 3 from geometric considerations and the assumption that the curvature of the bifurcation diagram is the same for both the constrained and unconstrained systems. Thus, it seems that the accuracy of the approximation depends upon the effect of the constraint on the curvature of the constrained system bifurcation diagram.

Potential application of the theory would benefit greatly from classification of which type of system typically exhibited which particular inaccuracy. In addition, since (2.24) represents a second order approximation of the relation of λ to θ, we would like also to know which inaccuracies might be corrected for by simply refining (2.24) to include higher order terms (see Appendix 5). In fact, the appropriate use of (2.24) most likely depends upon both the complete model being tested and the specific constraint within the model. In other words, (2.24) might prove to be an excellent approximation for one type of constraint but not another within the same model of a power system, or (2.24) might prove accurate for all constraints possible in one model, but not accurate for constraining the same quantities in a different model of the same system. We would like to know if the accuracy generally improves if λ appears only linearly or not at all in the dynamic equations for the variables to be constrained, or if the original equations are at most quadratic. Future mathematical analysis and numerical simulation should determine how accurate an approximation (2.24) is for each specific system, constraint, and model.

The practical usefulness of the theory presented in this thesis can be finally determined only by test and simulation with realistic power systems. However, the results may also find greater application if the theoretical basis on which it stands can be broadened. The analysis in this thesis should provide a firm basis for extending the results and testing them in the various applications to voltage collapse problems in power systems.

Appendix 1. The Schur Complement and Eigenvectors

We first show that the y–partitions of the zero eigenvectors of F_z are zero eigenvectors of the Schur complement of f_x. We write $\mathbf{w} = (\mathbf{w}^x, \mathbf{w}^y)$ and $\mathbf{v} = \begin{pmatrix} \mathbf{v}^x \\ \mathbf{v}^y \end{pmatrix}$ to represent the left and right eigenvectors of F_z. For F_z singular

$$\mathbf{w}F_z = 0, \quad F_z\mathbf{v} = 0 \tag{A1.1}$$

which leads to the component relations:

$$\mathbf{w}^x f_x + \mathbf{w}^y g_x = 0, \quad f_x\mathbf{v}^x + f_y\mathbf{v}^y = 0 \tag{A1.2}$$

$$\mathbf{w}^x f_y + \mathbf{w}^y g_y = 0, \quad g_x\mathbf{v}^x + g_y\mathbf{v}^y = 0 \tag{A1.3}$$

The Schur complement of the matrix f_x is the matrix

$$S = g_y - g_x[f_x]^{-1}f_y \tag{A1.4}$$

Multiplying S by $\mathbf{v}^y$ and applying the component relations leads to

$$S\mathbf{v}^y = g_y\mathbf{v}^y - g_x[f_x]^{-1}f_y\mathbf{v}^y = g_y\mathbf{v}^y + g_x\mathbf{v}^x = 0$$

so that $\mathbf{v}^y$ is a right zero eigenvector for S. Similarly, premultiplication by $\mathbf{w}^y$ leads to

$$\mathbf{w}^y S = \mathbf{w}^y g_y - \mathbf{w}^y g_x[f_x]^{-1}f_y = \mathbf{w}^y g_y + \mathbf{w}^x f_y = 0$$

and w^y is a left zero eigenvector for S.

Now we show that any zero eigenvector of S is collinear to the y partition of the zero eigenvector of F_z. Assume we know left and right zero eigenvectors of S so that

$$wS = 0, \quad Sv = 0 \tag{A1.5}$$

Then the component relations (A1.2) and (A1.3) imply that the vectors

$$(-wg_x[f_x]^{-1}, w) \qquad \begin{pmatrix} -[f_x]^{-1}f_y v \\ v \end{pmatrix}$$

are left and right zero eigenvectors of F_z. Since F_z has a unique simple zero eigenvalue, w and v must be proportional to $\mathbf{w}^y$ and $\mathbf{v}^y$ respectively, and thus S has a one dimensional kernel as does F_z. However, if $\mathbf{w}^y\mathbf{v}^y = 0$, then the left and right zero eigenvectors of S are orthogonal, implying that the algebraic multiplicity of the zero eigenvalue of S is greater than one and that S has generalized zero eigenvectors in addition to eigenvectors w and v. In short, if $\mathbf{w}^y\mathbf{v}^y \neq 0$ then S has a unique simple zero eigenvalue, and if $\mathbf{w}^y\mathbf{v}^y = 0$ then S has a unique non-simple zero eigenvalue.

Appendix 2. Derivatives of h, Θ, and H

The derivatives of h are derived by successively applying the chain rule to the implicit definition of h. For example,

$$0 = f\big(h(y,\lambda), y, \lambda\big) \Longrightarrow f_x h_y + f_y = 0$$

$$\Longrightarrow f_{xx} h_y h_y + 2 f_{xy} h_y + f_x h_{yy} + f_{yy} = 0$$

Since f_x is required to be invertible, we solve the above equations for h_y and h_{yy} to obtain

$$\begin{aligned} h_y &= -[f_x]^{-1} f_y \\ h_{yy} &= -[f_x]^{-1}(f_{xx} h_y h_y + 2 f_{xy} h_y + f_{yy}) \end{aligned} \tag{1.10}$$

Other derivatives of h are computed similarly,

$$\begin{aligned} h_\lambda &= -[f_x]^{-1} f_\lambda \\ h_{\lambda\lambda} &= -[f_x]^{-1}(f_{xx} h_\lambda h_\lambda + 2 f_{x\lambda} h_\lambda + f_{\lambda\lambda}) \\ h_{y\lambda} &= -[f_x]^{-1}(f_{xx} h_y h_\lambda + f_{x\lambda} h_y + f_{xy} h_\lambda + f_{y\lambda}) \\ h_{yyy} &= -[f_x]^{-1}(3 f_{xx} h_{yy} h_y + 3 f_{xy} h_{yy} + f_{xxx} h_y h_y h_y \\ &\quad + 3 f_{xxy} h_y h_y + 3 f_{xyy} h_y + f_{yyy}) \end{aligned}$$

In deriving the first term of h_{yyy}, note that the symmetry of the tensor f_{xx} implies that

$$f_{xx} h_{yy} h_y = f_{xx} h_y h_{yy}$$

The map Θ is defined by

$$\Theta(x,\theta) = x_* \theta + (1-\theta) x \tag{1.13}$$

Differentiation yields,

$$\begin{aligned} \Theta_x &= 1 - \theta \\ \Theta_\theta &= x_* - x \\ \Theta_{xx} &= 0 \\ \Theta_{\theta\theta} &= 0 \\ \Theta_{x\theta} &= -1 \end{aligned}$$

The map H is defined by:

$$H(y,\lambda,\theta) = g\big(\Theta(h(y,\lambda),\theta), y, \lambda\big) \tag{1.14}$$

We compute the derivatives of H with the chain rule. First order:

$$\begin{aligned}
H_y &= g_x\Theta_x h_y + g_y \\
H_\theta &= g_x\Theta_\theta \\
H_\lambda &= g_x\Theta_x h_\lambda + g_\lambda
\end{aligned}$$

Second order:

$$\begin{aligned}
H_{yy} &= g_{xx}(\Theta_x h_y)(\Theta_x h_y) + g_x\Theta_{xx}h_y h_y + 2g_{xy}\Theta_x h_y + g_x\Theta_x h_{yy} + g_{yy} \\
H_{\theta\theta} &= g_{xx}\Theta_\theta\Theta_\theta + g_x\Theta_{\theta\theta} \\
H_{\lambda\lambda} &= g_{xx}(\Theta_x h_\lambda)(\Theta_x h_\lambda) + g_x\Theta_{xx}h_\lambda h_\lambda + 2g_{x\lambda}\Theta_x h_\lambda + g_x\Theta_x h_{\lambda\lambda} + g_{\lambda\lambda} \\
H_{y\theta} &= g_{xx}\Theta_\theta(\Theta_x h_y) + g_x\Theta_{x\theta}h_y + g_{xy}\Theta_\theta \\
H_{y\lambda} &= g_{xx}(\Theta_x h_y)(\Theta_x h_\lambda) + g_{xy}\Theta_x h_\lambda + g_x\Theta_x h_{y\lambda} + g_x\Theta_{xx}h_y h_\lambda + g_{x\lambda}\Theta_x h_y + g_{y\lambda} \\
H_{\lambda\theta} &= g_{x\lambda}\Theta_\theta + g_{xx}(\Theta_x h_\lambda)\Theta_\theta + g_x\Theta_{x\theta}h_\lambda
\end{aligned}$$

Third order needed to calculate $\Lambda_{\theta\theta\theta}$:

$$\begin{aligned}
H_{yyy} = {} & g_{xxx}(\Theta_x h_y)(\Theta_x h_y)(\Theta_x h_y) + 3g_{xxy}(\Theta_x h_y)(\Theta_x h_y) \\
& + 3g_{xx}(\Theta_{xx}h_y h_y)(\Theta_x h_y) + 3g_{xx}(\Theta_x h_y)(\Theta_x h_{yy}) \\
& + g_x\Theta_{xxx}h_y h_y h_y + 3g_x\Theta_{xx}h_y h_{yy} \\
& + 3g_{xyy}\Theta_x h_y + 3g_{xy}\Theta_{xx}h_y h_y + 3g_{xy}\Theta_x h_{yy} \\
& + g_x\Theta_x h_{yyy} + g_{yyy} \\
H_{yy\theta} = {} & g_{xxx}\Theta_\theta(\Theta_x h_y)(\Theta_x h_y) + 2g_{xx}(\Theta_{x\theta}h_y)(\Theta_x h_y) + g_{xx}\Theta_\theta(\Theta_{xx}h_y h_y) \\
& + g_x\Theta_{xx\theta}h_y h_y + 2g_{xxy}\Theta_\theta(\Theta_x h_y) + 2g_{xy}\Theta_{x\theta}h_y \\
& + g_{xx}\Theta_\theta(\Theta_x h_{yy}) + g_x\Theta_{x\theta}h_{yy} + g_{xyy}\Theta_\theta
\end{aligned}$$

We evaluate the derivatives of Θ at (x_*,θ_*), where $\theta_* = 0$.

$$\begin{aligned}
\Theta_x &= 1 \\
\Theta_\theta &= 0 \\
\Theta_{xx} &= 0 \\
\Theta_{\theta\theta} &= 0 \\
\Theta_{x\theta} &= -1
\end{aligned}$$

Substitution into the expressions for the derivatives of H, evaluated at the bifurcation $(y_*, \lambda_*, \theta_*)$ yields,

First order:

$$\begin{aligned} H_y &= g_x h_y + g_y \\ H_\theta &= 0 \\ H_\lambda &= g_x h_\lambda + g_\lambda \end{aligned}$$

Second order:

$$\begin{aligned} H_{yy} &= g_{xx} h_y h_y + 2 g_{xy} h_y + g_x h_{yy} + g_{yy} \\ H_{\theta\theta} &= 0 \\ H_{\lambda\lambda} &= g_{xx} h_\lambda h_\lambda + 2 g_{x\lambda} h_\lambda + g_x h_{\lambda\lambda} + g_{\lambda\lambda} \\ H_{y\theta} &= -g_x h_y = g_x [f_x]^{-1} f_y \\ H_{y\lambda} &= g_{xx} h_y h_\lambda + g_{xy} h_\lambda + g_x h_{y\lambda} + g_{x\lambda} h_y + g_{y\lambda} \\ H_{\lambda\theta} &= -g_x h_\lambda \end{aligned}$$

Third order needed to calculate $\Lambda_{\theta\theta\theta}$:

$$\begin{aligned} H_{yyy} &= g_{xxx} h_y h_y h_y + 3 g_{xxy} h_y h_y + 3 g_{xx} h_y h_{yy} \\ &\quad + 3 g_{xyy} h_y + 3 g_{xy} h_{yy} + g_x h_{yyy} + g_{yyy} \\ H_{yy\theta} &= -2 g_{xx} h_y h_y - 2 g_{xy} h_y - g_x h_{yy} \end{aligned}$$

Appendix 3. $wH_{yy}(v,v)$, wH_λ, and $wH_{y\theta}v$ at the bifurcation

We derive expressions for $wH_{yy}(v,v)$, wH_λ, and $wH_{y\theta}v$ at the bifurcation in terms of F in (1.1) and (1.2), and show that the bifurcation of (1.16) satisfies the transversality conditions for the saddle node bifurcation

$$wH_{yy}(v,v) \neq 0 \tag{1.17}$$

$$wH_\lambda \neq 0 \tag{1.18}$$

In Appendix 1 we show that the y–partitions of the zero eigenvectors $\mathbf{w}$ and $\mathbf{v}$ of F_z are zero eigenvectors for the Schur complement of f_x. Since at the bifurcation H_y is the Schur complement of f_x, as zero eigenvectors for H_y we use:

$$v = \mathbf{v}^y$$

$$w = \mathbf{w}^y$$

where $\mathbf{w}^y$ and $\mathbf{v}^y$ are the y–partitions of $\mathbf{w}$ and $\mathbf{v}$ respectively. Note that we do not yet assume that w and v are normalized. We now verify the transversality condition (1.18) using the component relations (A1.2) and (A1.3) and applying (1.4)

$$\begin{aligned} wH_\lambda &= \mathbf{w}^y\big(g_\lambda - g_x[f_x]^{-1}f_\lambda\big) = \mathbf{w}^y g_\lambda - \mathbf{w}^y g_x[f_x]^{-1}f_\lambda \\ &= \mathbf{w}^y g_\lambda + \mathbf{w}^x f_x[f_x]^{-1}f_\lambda = \mathbf{w}^y g_\lambda + \mathbf{w}^x f_\lambda = \mathbf{w}F_\lambda \end{aligned}$$

Thus,

$$\mathbf{w}F_\lambda \neq 0 \tag{1.4}$$

implies (1.18).

Similarly, using the derivations for h_y and h_{yy} from Appendix 2 and noting that the component relations (A1.2) and (A1.3) can be expressed as

$$h_y\mathbf{v}^y = \mathbf{v}^x, \qquad \mathbf{w}^y g_x = -\mathbf{w}^x f_x$$

we verify that (1.3) implies transversality condition (1.17) as follows

$$\begin{aligned} wH_{yy}(v,v) &= \mathbf{w}^y\big(g_{xx}h_yh_y + 2g_{xy}h_y + g_xh_{yy} + g_{yy}\big)(\mathbf{v}^y,\mathbf{v}^y) \\ &= \mathbf{w}^y g_{xx}(h_y\mathbf{v}^y, h_y\mathbf{v}^y) + 2\mathbf{w}^y g_{xy}(h_y\mathbf{v}^y,\mathbf{v}^y) + \mathbf{w}^y g_x(h_{yy}\mathbf{v}^y,\mathbf{v}^y) + \mathbf{w}^y g_{yy}(\mathbf{v}^y,\mathbf{v}^y) \\ &= \mathbf{w}^y g_{xx}(\mathbf{v}^x,\mathbf{v}^x) + 2\mathbf{w}^y g_{xy}(\mathbf{v}^x,\mathbf{v}^y) - \mathbf{w}^x f_x(h_{yy}\mathbf{v}^y,\mathbf{v}^y) + \mathbf{w}^y g_{yy}(\mathbf{v}^y,\mathbf{v}^y) \\ &= \mathbf{w}^y g_{xx}(\mathbf{v}^x,\mathbf{v}^x) + 2\mathbf{w}^y g_{xy}(\mathbf{v}^x,\mathbf{v}^y) \\ &\quad + \mathbf{w}^x[f_{xx}h_yh_y + 2f_{xy}h_y + f_{yy}](\mathbf{v}^y,\mathbf{v}^y) + \mathbf{w}^y g_{yy}(\mathbf{v}^y,\mathbf{v}^y) \\ &= \mathbf{w}^y g_{xx}(\mathbf{v}^x,\mathbf{v}^x) + 2\mathbf{w}^y g_{xy}(\mathbf{v}^x,\mathbf{v}^y) \\ &\quad + \mathbf{w}^x f_{xx}(\mathbf{v}^x,\mathbf{v}^x) + 2\mathbf{w}^x f_{xy}(\mathbf{v}^x,\mathbf{v}^y) + \mathbf{w}^x f_{yy}(\mathbf{v}^y,\mathbf{v}^y) + \mathbf{w}^y g_{yy}(\mathbf{v}^y,\mathbf{v}^y) \\ &= \mathbf{w}F_{zz}(\mathbf{v},\mathbf{v}) \end{aligned}$$

Thus,

$$\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v}) \neq 0 \tag{1.3}$$

implies (1.17). Note that the terms $\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})$ and $wH_{yy}(v,v)$ are related to the derivatives of the eigenvalues of F_z and H_y as follows.

We now select a normalization for the eigenvectors of H_y. In particular, we assume that H_y has a unique simple zero eigenvalue at $(y_*, \lambda_*, \theta_*)$, so that $\mathbf{w}^y\mathbf{v}^y \neq 0$ (see Appendix 1) and we choose normalized left and right zero eigenvectors

$$v = \mathbf{v}^y$$
$$w = \frac{\mathbf{w}^y}{\mathbf{w}^y\mathbf{v}^y}$$

so that $wv = 1$ The simple eigenvalues and eigenvectors of a matrix are smooth functions of the entries of the matrix. In the case of the Jacobian matrix of a smooth function of n variables, the eigenvalues and eigenvectors are smooth functions of the entries of the Jacobian matrix which are smooth functions of the n variables. We write $\boldsymbol{\mu}(z)$ and $\mu(y)$ to represent the respective eigenvalues of F_z and H_y in a neighborhood of the origin with $\boldsymbol{\mu}(z_*) = 0$ and $\mu(y_*) = 0$. Thus,

$$\boldsymbol{\mu}(z) = \mathbf{w}(z)F_z(z)\mathbf{v}(z), \qquad \mu(y) = w(y)H_y(y)v(y)$$

The direction that the stable and unstable solutions approach each other at a saddle node bifurcation is the right zero eigenvector of the Jacobian matrix at the bifurcation[4]. Thus, the directional derivative of the zero eigenvalue in the direction of the right zero eigenvector represents the rate at which the zero eigenvalue moves as the bifurcation occurs. We write $\hat{\mathbf{v}}$ and $\hat{v}$ to represent unit vectors in the $\mathbf{v}$ and v directions. By hypothesis in Chapter 1, we select $\mathbf{w}$ and $\mathbf{v}$ so that $\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})$ is positive. Evaluated at the bifurcation

$$D_z\left(\boldsymbol{\mu}(z)\right)\hat{\mathbf{v}} = \boldsymbol{\mu}_z\hat{\mathbf{v}} = \frac{\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})}{|\mathbf{v}|} \tag{$A3.1$}$$

and

$$D_y\left(\mu(y)\right)\hat{v} = \mu_y\hat{v} = \frac{wH_{yy}(v,v)}{|v|} = \frac{\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})}{\mathbf{w}^y\mathbf{v}^y\ |\mathbf{v}^y|} \tag{$A3.2$}$$

The sign of $\mu_y\hat{v}$ has the same sign as $\boldsymbol{\mu}_z\hat{\mathbf{v}}$ if $\mathbf{w}^y\mathbf{v}^y$ is positive. Thus, for $\mathbf{w}^y\mathbf{v}^y$ positive, the critical eigenvalue of H_y crosses the origin in the same direction as the critical eigenvalue of F_z, and for negative $\mathbf{w}^y\mathbf{v}^y$ the critical eigenvalue of H_y crosses the origin in the opposite direction as the critical eigenvalue of F_z. Since we assume that the system described by (1.1) is stable before the bifurcation, we know that the critical eigenvalue of F_z moves from negative to positive at the bifurcation.

We now examine the sensitivity to θ of the zero eigenvalue of H_y at the bifurcation $(y_*, \lambda_*, \theta_*)$. Substitution for v and w and the expression for $H_{y\theta}$ from Appendix 2 leads to derivation of (2.3) from (2.2),

$$\mu_\theta = wH_{y\theta}v \tag{2.2}$$

$$wH_{y\theta}v = \frac{1}{\mathbf{w}^y\mathbf{v}^y}\mathbf{w}^y(-g_x h_y)\mathbf{v}^y = \frac{\mathbf{w}^y(-g_x\mathbf{v}^x)}{\mathbf{w}^y\mathbf{v}^y} = \frac{\mathbf{w}^x f_x \mathbf{v}^x}{\mathbf{w}^y\mathbf{v}^y} \tag{2.3}$$

As the constraint is applied, the zero eigenvalue of H_y moves in the direction determined by the sign of (2.3). However, we know from (A3.1) and (A3.2) that the sign of $\mathbf{w}^y\mathbf{v}^y$ determines the relation between the direction of the zero eigenvalue of F_z and the direction of the zero eigenvalue of H_y. Thus, the sign of $\mathbf{w}^x f_x \mathbf{v}^x$ determines the direction the zero eigenvalue of F_z moves as the constraint is applied. For the constraint to be stabilizing $\mathbf{w}^x f_x \mathbf{v}^x$ must be negative. However, the constrained system may still be unstable even if $\mathbf{w}^x f_x \mathbf{v}^x$ is negative since other eigenvalues of $g_y|_{(y_*,\lambda_*,\theta_*)}$ may have positive real parts. The sign of $\mathbf{w}^x f_x \mathbf{v}^x$ determines the sign of the real eigenvalue closest to the origin after encountering the constraint; the constrained system is stable only if all the eigenvalues of $g_y|_{(y_*,\lambda_*,\theta_*)}$ have negative real parts.

Substitution of the previous expressions (2.2) and (2.3) into (2.20) leads to derivation of (2.21),

$$\Lambda_{\theta\theta} = \frac{(wH_{y\theta}v)^2}{wH_{yy}(v,v)wH_\lambda} \tag{2.20}$$

$$\Lambda_{\theta\theta} = \frac{(wH_{y\theta}v)^2}{wH_{yy}(v,v)wH_\lambda} = \frac{\left(\frac{\mathbf{w}^x f_x \mathbf{v}^x}{\mathbf{w}^y\mathbf{v}^y}\right)^2}{\frac{\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})\mathbf{w}F_\lambda}{(\mathbf{w}^y\mathbf{v}^y)^2}} = \frac{(\mathbf{w}^x f_x \mathbf{v}^x)^2}{\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})\ \mathbf{w}F_\lambda} \tag{2.21}$$

Note that although we assumed that w and v were normalized so that $wv = 1$, evaluation of (2.20) is invariant to normalization. The derivation only requires that w and v be y–partitions of any left and right zero eigenvectors of F_z.

Appendix 4. (U_y, U_λ) is Invertible at the Bifurcation

The map U is defined by

$$U(y,\lambda,\theta) = \begin{pmatrix} H(y,\lambda,\theta) \\ \mu(y,\lambda,\theta) \end{pmatrix} \tag{2.5}$$

We proceed to show that (U_y, U_λ) is invertible at $(y_*, \lambda_*, \theta_*)$.

$$(U_y, U_\lambda) = \begin{pmatrix} H_y & H_\lambda \\ \mu_y & \mu_\lambda \end{pmatrix} = \begin{pmatrix} H_y & H_\lambda \\ wH_{yy}v & wH_{y\lambda}v \end{pmatrix}$$

where H_y is a $n-s \times n-s$ matrix; $wH_{yy}v$, is a $n-s$ row vector; H_λ, is a $n-s$ column vector; and $wH_{y\lambda}v$ is a scalar. (U_y, U_λ) is invertible if

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is the only solution to

$$(U_y, U_\lambda)\begin{pmatrix} a \\ b \end{pmatrix} = 0 \tag{$A4.1$}$$

Note that a is a $n-s$ column vector and b is a scalar.

The vector H_λ is not in the range of H_y since $wH_\lambda \neq 0$ by transversality condition (1.18). Thus, to satisfy the first $n-s$ rows of (A4.1), $b = 0$ and a must be collinear to v. Let $a = cv$ for some scalar c. Then the remaining row equation requires that $cwH_{yy}(v,v) = 0$, which implies either $c = 0$ or $wH_{yy}(v,v) = 0$. But $wH_{yy}(v,v) = 0$ violates transversality condition (1.17), so a must also be a zero vector, and thus (U_y, U_λ) is invertible at the bifurcation.

Appendix 5. Computation of $\Lambda_{\theta\theta\theta}$

Substitution into (2.13) of the expressions for Y_θ and $\Lambda_{\theta\theta}$ from (2.18) and (2.20) and evaluating at the bifurcation ($H_{\theta\theta} = 0$, $\Lambda_\theta = 0$) leads to

$$\begin{aligned} H_y Y_{\theta\theta} =& \frac{-(wH_{y\theta}v)^2}{(wH_{yy}(v,v))^2} H_{yy}(v,v) \\ &+ \frac{2(wH_{y\theta}v)}{wH_{yy}(v,v)} H_{y\theta}v - \frac{(wH_{y\theta}v)^2}{wH_{yy}(v,v)wH_\lambda} H_\lambda \end{aligned} \tag{A5.1}$$

Differentiation of (2.13), and removing terms $H_{\theta\theta\theta}$, $H_{y\theta\theta}$, Λ_θ that evaluate to zero at the bifurcation gives

$$\begin{aligned} 0 =& H_{yyy}Y_\theta Y_\theta Y_\theta + 3H_{yy\theta}Y_\theta Y_\theta + 3H_{yy}Y_\theta Y_{\theta\theta} \; + 3H_{y\theta}Y_{\theta\theta} \\ &+ H_\lambda \Lambda_{\theta\theta\theta} + 3H_{y\lambda}Y_\theta\Lambda_{\theta\theta} + H_y Y_{\theta\theta\theta} + 3H_{\lambda\theta}\Lambda_{\theta\theta} \end{aligned} \tag{A5.2}$$

Since H_y has a unique zero eigenvalue, from (A5.1) we are able to find all the components of $Y_{\theta\theta}$ except that in the v direction. However, since $Y_{\theta\theta}$ is only needed for substitution into (A5.2), where eventually premultiplication by w leads to an equation for $\Lambda_{\theta\theta\theta}$ and obliterates the v component of $Y_{\theta\theta}$ anyway, it is sufficient to solve for $Y_{\theta\theta}(mod\ v)$ from (A5.1) alone.

Once $Y_{\theta\theta}(mod\ v)$ is known, substitution into (A5.2) and premultiplication by w leads to solution for $\Lambda_{\theta\theta\theta}$,

$$\begin{aligned} \Lambda_{\theta\theta\theta} =& \frac{wH_{yyy}(v,v,v)}{wH_\lambda}\left(\frac{wH_{y\theta}v}{wH_{yy}(v,v)}\right)^3 - \frac{wH_{yy\theta}(v,v)}{wH_\lambda}\left(\frac{wH_{y\theta}v}{wH_{yy}(v,v)}\right)^2 \\ &+ wH_{y\lambda}v \; wH_{y\theta}v\left(\frac{wH_{y\theta}v}{wH_{yy}(v,v)\; wH_\lambda}\right)^2 - \frac{wH_{\lambda\theta}(wH_{y\theta}v)^2}{wH_{yy}(v,v)\; wH_\lambda} \\ &+ 3\left[\left(\frac{wH_{y\theta}v\; wH_{yy}v}{wH_{yy}(v,v)}\right) - wH_{y\theta}\right]Y_{\theta\theta} \end{aligned} \tag{A5.3}$$

Which alternatively may be expressed as:

$$\begin{aligned} \Lambda_{\theta\theta\theta} =& (wH_{y\theta}v)^3\left[\frac{wH_{yyy}(v,v,v)}{(wH_{yy}(v,v))^3 wH_\lambda} + \frac{wH_{y\lambda}v}{(wH_{yy}(v,v)wH_\lambda)^2}\right] \\ &- (wH_{y\theta}v)^2\left[\frac{wH_{yy\theta}(v,v) + wH_{\lambda\theta}}{(wH_{yy}(v,v))^3 wH_\lambda}\right] \\ &+ 3(wH_{y\theta}v)\left[\frac{wH_{yy}v}{wH_{yy}(v,v)} - \frac{wH_{y\theta}}{wH_{y\theta}v}\right]Y_{\theta\theta} \end{aligned} \tag{2.22}$$

Appendix 6. Lyapunov–Schmidt Reduction

Roughly speaking, the generic saddle node bifurcation may be considered to occur along only one direction of the multidimensional space. The solutions of (1.2) near the bifurcation may be put in one to one correspondence with solutions of a reduced system with only one state variable. The state variable of the reduced system, denoted s, represents the projection of the equilibrium position onto a subspace tangent to the center manifold at the bifurcation (z_*, λ_*). At the bifurcation, $s = 0$. Equilibria of the system are the (s, λ) that satisfy

$$L(s, \lambda) = 0 \tag{3.1)(A6.1}$$

where,

$$L(s, \lambda) = \mathbf{w} F\big(\xi(s, \lambda), \lambda\big) \tag{A6.2}$$

and ξ is an injective map from the projected space to the original coordinates, implicitly defined by the equation

$$\rho_w F(\xi(s, \lambda), \lambda) = 0 \tag{A6.3}$$

with the decomposition

$$\xi(s, \lambda) = s\mathbf{v} + \rho_v \xi(s, \lambda) \tag{A6.4}$$

The map $\rho_v : \mathbb{R}^n \to \mathbb{R}^n$ is the projection orthogonal to the right eigenvector $\mathbf{v}$ at the bifurcation, and $\rho_w : \mathbb{R}^n \to \mathbb{R}^{n-1}$ represents the projection into the hyperplane orthogonal to the left eigenvector $\mathbf{w}$ at the bifurcation. The first derivatives of L,

$$L_s = \mathbf{w} F_z \xi_s, \qquad L_\lambda = \mathbf{w} F_z \xi_\lambda + \mathbf{w} F_\lambda$$

evaluated at the bifurcation are,

$$L_s = 0, \qquad L_\lambda = \mathbf{w} F_\lambda \tag{3.5}$$

The solutions of (A6.1) in the $s - \lambda$ plane are the bifurcation diagram of the system. Since by the transversality condition (1.4), L_λ is not zero at the bifurcation, by the implicit function theorem there is a neighborhood about the bifurcation where we may express λ as a function of s. Let σ represent that map, then in a neighborhood of the bifurcation

$$L\big(s, \sigma(s)\big) = 0 \tag{3.4)(A6.5}$$

Differentiating (A6.5) leads to the following equations that describe the derivatives of σ,

$$L_s + L_\lambda \sigma_s = 0 \Longrightarrow \sigma_s = 0$$

$$L_{ss} + 2L_{s\lambda}\sigma_s + L_\lambda \sigma_{ss} = 0 \Longrightarrow \sigma_{ss} = -\frac{L_{ss}}{L_\lambda}$$

We approximate the relation of λ to s on the bifurcation diagram near the bifurcation by a second order Taylor series,

$$\lambda = \sigma(0) + \sigma_s s + \frac{1}{2}\sigma_{ss}s^2 = -\frac{1}{2}\frac{L_{ss}}{L_\lambda}s^2 \tag{A6.6}$$

The second derivative of L with respect to s,

$$L_{ss} = \mathbf{w}F_{zz}\xi_s\xi_s + \mathbf{w}F_z\xi_{ss}$$

evaluated at the bifurcation is

$$L_{ss} = \mathbf{w}F_{zz}\xi_s\xi_s \tag{A6.7}$$

Differentiating (A6.3) and (A6.4) with respect to s at the bifurcation yields

$$\rho_w F_z \xi_s = 0, \qquad \xi_s = \mathbf{v} + \rho_v \xi_s$$

implying that

$$\xi_s = \mathbf{v}$$

Thus,

$$L_{ss} = \mathbf{w}F_{zz}(\mathbf{v},\mathbf{v}) \tag{3.5}$$

and (A6.5) may be written

$$\lambda = -\frac{1}{2}\frac{\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})}{\mathbf{w}F_\lambda}s^2 \tag{A6.8}$$

We parameterize the solutions to (A6.1) with the map

$$\alpha(s) = \big(s, \sigma(s)\big)$$

and compute the gradient, velocity, and acceleration vectors of the parameterization,

$$DL(s) = (L_s, L_\lambda), \qquad \dot\alpha(s) = (1, \sigma_s), \qquad \ddot\alpha(s) = (0, \sigma_{ss})$$

We use the formula for the curvature, κ, of a plane curve[26] to find the curvature of the bifurcation diagram,

$$\kappa(s) = \frac{\ddot{\alpha}(s)(DL(s))^T}{||\dot{\alpha}(s)||^2}$$

$$= \frac{\sigma_{ss} L_\lambda}{1 + \sigma_s \sigma_s}$$

Which, evaluated at the bifurcation gives,

$$\kappa = -L_{ss} = -\mathbf{w} F_{zz}(\mathbf{v}, \mathbf{v}) \tag{3.6}$$

The negative sign is a consequence of our sign convention that solutions exist only for negative λ.

Note that a Lyapunov-Schmidt reduction can be performed on the saddle node bifurcation in y and λ of (1.16) at $(y_*, \lambda_*, \theta_*)$ for θ constant at θ_*.

$$0 = H(y, \lambda, \theta_*) \tag{1.16}$$

The reduction follows the above derivation identically, with the eigenvectors of H_y choosen as in Appendix 3,

$$v = \mathbf{v}^y$$

$$w = \frac{\mathbf{w}^y}{\mathbf{w}^y \mathbf{v}^y}$$

where $\mathbf{w}^y$ and $\mathbf{v}^y$ are the y-partitions of $\mathbf{w}$ and $\mathbf{v}$ respectively. Note that the projection onto the right eigenvector of H_y is the same as projection onto the y-components of the right eigenvector of F_z. The corresponding derivatives of the new reduction L^H are

$$L^H_{ss} = \frac{L_{ss}}{\mathbf{w}^y \mathbf{v}^y} = \frac{\mathbf{w} F_{zz}(\mathbf{v}, \mathbf{v})}{\mathbf{w}^y \mathbf{v}^y}, \qquad L^H_\lambda = \frac{L_\lambda}{\mathbf{w}^y \mathbf{v}^y} = \frac{\mathbf{w} F_\lambda}{\mathbf{w}^y \mathbf{v}^y}$$

The saddle node bifurcation diagram of (1.16) is identical to that of (1.2), and (A6.6) and (3.5) are valid for both the saddle node bifurcation of (1.2) at (z_*, λ_*) and the saddle node bifurcation of (1.16) at $(y_*, \lambda_*, \theta_*)$ for constant θ. However, as noted in Appendix 3 the sign of $\mathbf{w}^y \mathbf{v}^y$ indicates if the change in stability of F at the bifurcation has the same orientation as the change in stability of H at the bifurcation. In other words, if $\mathbf{w}^y \mathbf{v}^y$ is negative, then the saddle node bifurcation diagram of (1.16) is identical to that of (1.2) but the stable portion of the diagram of (1.2) corresponds to the unstable portion of the diagram of (1.16).

Appendix 7. Transcritical Bifurcation at $(y_*, \lambda_*, \theta_*)$

The transversality conditions for the transcritical bifurcation[12] are that evaluated at the bifurcation $(y_*, \lambda_*, \theta_*)$, H_y has a unique simple zero eigenvalue and

$$wH_\theta = 0 \tag{A7.1}$$

$$wH_{y\theta}v \neq 0 \tag{A7.2}$$

$$wH_{yy}(v, v) \neq 0 \tag{A7.3}$$

By hypothesis F_z has a unique simple zero eigenvalue at the bifurcation and in Appendix 1 we show that the generic assumption $\mathbf{w}^y\mathbf{v}^y \neq 0$ assures that H_y also has a unique simple zero eigenvalue at the bifurcation. In Appendix 2 we show that at the bifurcation $H_\theta = 0$ satisfying (A7.1). (A7.3) is satisfied by (1.17), as shown in Appendix 3. Condition (A7.2) is satisfied when (2.26) does not evaluate to zero.

$$\mu_\theta = wH_{y\theta}v = \frac{\mathbf{w}^x f_x \mathbf{v}^x}{\mathbf{w}^y\mathbf{v}^y} \tag{2.26}$$

Since by assumption $\mathbf{w}^y\mathbf{v}^y \neq 0$, $wH_{y\theta}v \neq 0$ implies that $\mathbf{w}^x f_x \mathbf{v}^x \neq 0$. Assume $\mathbf{w}^x f_x \mathbf{v}^x = 0$, then the vectors $(\mathbf{w}^x, 0)$ and $\begin{pmatrix} \mathbf{v}^x \\ 0 \end{pmatrix}$ are left and right zero eigenvectors for $F_z|_{(z_*,\lambda_*)}$ violating either the hypothesis that $\mathbf{w}^y\mathbf{v}^y \neq 0$ or that $F_z|_{(z_*,\lambda_*)}$ has a unique simple zero eigenvalue. So (A7.1) is implied by the hypothesis and

$$0 = H(y, \lambda_*, \theta) \tag{1.16}$$

exhibits a generic transcritical bifurcation at $(y_*, \lambda_*, \theta_*)$. We apply Lyapunov–Schmidt reduction to the transcritical bifurcation in y and θ, considering λ constant at λ_*. The solutions for λ constant at λ_* of (1.16) near the bifurcation may be put in one to one correspondence with solutions of a reduced system with only one state variable. The condensed state variable, denoted s, represents the projection onto the right eigenvector at the bifurcation (y_*, θ_*). At the bifurcation, $s = 0$. Equilibria of the system are the (s, θ) that satisfy

$$L^t(s, \theta) = 0 \tag{A7.3}$$

where,

$$L^t(s, \theta) = wH\big(\xi(s, \theta), \theta\big) \tag{A7.4}$$

and the map $\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n-1} \times \mathbb{R}$ is implicitly defined by the equation

$$\rho_w H(\xi(s, \theta), \theta) = 0 \tag{A7.5}$$

with the decomposition

$$\xi(s,\theta) = sv + \rho_v \xi(s,\theta) \tag{A7.6}$$

The map $\rho_v : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is the projection orthogonal to the right eigenvector v at the bifurcation, and $\rho_w : \mathbb{R}^{n-1} \to \mathbb{R}^{n-2}$ is the projection onto the hyperplane orthogonal to the left eigenvector w at the bifurcation. Note that ρ_v for the reduction of the saddle node bifurcation of (1.16) in Appendix 6 is identical to the ρ_v for the reduction of the transcritical bifurcation of (1.16). Since the reduction of the saddle node bifurcation of (1.16) is equivalent to the reduction of the saddle node bifurcation of (1.2) (see Appendix 6), we can associate the distance s on the transcritical bifurcation diagram of (1.16) with the distance s of the saddle node bifurcation diagram of (1.2). The first derivatives of L^t,

$$L^t_s = wH_y\xi_s, \qquad L^t_\theta = wH_y\xi_\theta + wH_\theta$$

evaluated at the bifurcation (see Appendix 2) are,

$$L^t_s = 0, \qquad L^t_\theta = 0$$

The second derivatives of L^t are,

$$\begin{aligned} L^t_{ss} &= wH_{yy}\xi_s\xi_s \\ L^t_{s\theta} &= wH_{yy}\xi_\theta\xi_s + wH_{y\theta}\xi_s + wH_y\xi_{s\theta} \\ L^t_{\theta\theta} &= wH_{yy}\xi_\theta\xi_\theta + 2wH_{y\theta}\xi_\theta + wH_{\theta\theta} \end{aligned}$$

Differentiating (A7.5) and (A7.6) with respect to s at the bifurcation yields

$$\rho_w H_y \xi_s = 0, \qquad \xi_s = v + \rho_v\xi_s$$

implying that

$$\xi_s = v$$

Differentiating (A7.5) and (A7.6) with respect to θ at the bifurcation yields

$$\rho_w(H_y\xi_\theta + H_\theta) = 0, \qquad \xi_\theta = \rho_v\xi_\theta$$

which, since $H_\theta = 0$ (see Appendix 3) at the bifurcation, implies that

$$\xi_\theta = 0$$

Hence, evaluated at the bifurcation,

$$L_{ss}^t = wH_{yy}(v,v) = \frac{\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})}{\mathbf{w}^y\mathbf{v}^y}, \qquad L_{s\theta}^t = wH_{y\theta}v = \frac{\mathbf{w}^x f_x \mathbf{v}^x}{\mathbf{w}^y\mathbf{v}^y}, \qquad L_{\theta\theta}^t = 0$$

The solution of (A7.3) in the s-θ plane is the transcritical bifurcation diagram in s and θ. We approximate the relation of θ to s on the bifurcation diagram near the bifurcation by a second order Taylor series,

$$L^t(s,\theta) = L^t(0,0) + L_s^t s + L_\theta^t \theta + L_{s\theta}^t s\theta + \frac{1}{2}L_{ss}^t s^2 + \frac{1}{2}L_{\theta\theta}^t \theta^2 = 0$$

which reduces to

$$L^t(s,\theta) = s\left(L_{s\theta}^t\theta + \frac{1}{2}L_{ss}^t s\right) = 0 \tag{A7.7}$$

The solution is the line $s = 0$ and the line through the origin with slope

$$\frac{\Delta s}{\Delta\theta} = -2\frac{L_{s\theta}^t}{L_{ss}^t} = -2\frac{\mathbf{w}^x f_x \mathbf{v}^x}{\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v})}$$

Appendix 8. Examples

In this section we present several examples to illustrate the application of the calculations previously derived. We choose test systems for which the actual margin change can be calculated, and hence the accuracy of the approximation determined.

The first example involves a dynamical system with two degrees of freedom and a one dimensional parameter,

$$F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$$

$$\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F(z, \lambda) = \begin{pmatrix} f(x, y, \lambda) \\ g(x, y, \lambda) \end{pmatrix} = \begin{pmatrix} y - x \\ x - y^2 - \frac{1}{4} - \lambda \end{pmatrix}$$

We study the effect of constraining the x variable at the bifurcation. We compute the derivative of F with respect to z,

$$F_z = \begin{pmatrix} -1 & 1 \\ 1 & -2y \end{pmatrix}$$

The saddle node bifurcation occurs at $(x_*, y_*, \lambda_*) = (\frac{1}{2}, \frac{1}{2}, 0)$. The Jacobian matrix has eigenvalues of 0 and -2 at the saddle node bifurcation. Normalized left and right eigenvectors corresponding to the zero eigenvalue are

$$w = -\left(\frac{1}{2} \quad , \quad \frac{1}{2} \right) \quad v = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The second derivative of F with respect to z is,

$$F_{zz} = \begin{pmatrix} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \\ \begin{pmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{pmatrix} \end{pmatrix} = -\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \end{pmatrix}$$

and the derivative of F with respect to λ is

$$F_\lambda = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

We evaluate (2.24) to determine the stability margin of the constrained system,

$$\lambda_{c*} = \frac{(\mathbf{w}^x f_x \mathbf{v}^x)^2}{2\mathbf{w}F_{zz}(\mathbf{v}, \mathbf{v})\ \mathbf{w}F_\lambda} \tag{2.24}$$

$$= \frac{(\frac{-1}{2})^2}{2(1)\ (\frac{1}{2})} = \frac{1}{4}$$

Thus, we expect the bifurcation of the constrained system to occur at $\lambda_* = \frac{1}{4}$. Since the constrained system will have only one eigenvalue and $\mathbf{w}^x f_x \mathbf{v}^x$ is negative, we expect the constraint to be stabilizing, that is, the eigenvalue of the constrained system should be negative at (y_*, λ_*).

The constrained system differential equation is

$$\dot{y} = g(x_*, y, \lambda) = x_* - y^2 - \frac{1}{4} - \lambda = \frac{1}{2} - y^2 - \frac{1}{4} - \lambda = -y^2 + \frac{1}{4} - \lambda \qquad (A8.1)$$

The eigenvalue of the constrained system is

$$g_y = -2y \qquad (A8.2)$$

which evaluated at (y_*, λ_*) is

$$g_y|_* = -1$$

Thus, the constrained system is stable at (y_*, λ_*). The bifurcation point of the constrained system is the pair (y_{c*}, λ_{c*}) such that (A8.1) and (A8.2) evaluate to zero. The bifurcation point of the constrained system is thus

$$(y_{c*}, \lambda_{c*}) = (\, 0 \quad , \quad \tfrac{1}{4}\,)$$

For this simple system the approximation (2.24) is exact.

The next example illustrates constraint of multiple variables at a saddle node bifurcation.

$$F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$$

$$\dot{z} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y} \end{pmatrix} = F(z, \lambda) = \begin{pmatrix} f(x, y, \lambda) \\ g(x, y, \lambda) \end{pmatrix} = \begin{pmatrix} x_2 \\ -2x_1 - x_2 + y \\ -x_1 - y^2 - \frac{1}{16} - \lambda \end{pmatrix}$$

We study the effect of constraining the x variables at the bifurcation. We compute the derivative of F with respect to z,

$$F_z = \begin{pmatrix} 0 & 1 & 0 \\ -2 & -1 & 1 \\ -1 & 0 & -2y \end{pmatrix}$$

The saddle node bifurcation occurs at $(x_{1_*}, x_{2_*}, y_*, \lambda_*) = (-\frac{1}{8}, 0, -\frac{1}{4}, 0)$. The Jacobian matrix has eigenvalues of 0 and $-\frac{1}{4} \pm \frac{\sqrt{23}}{4} i$ at the saddle node bifurcation. Normalized left and right eigenvectors corresponding to the zero eigenvalue are

$$w = (\, 1 \quad , \quad 1 \quad , \quad -2\,), \qquad v = \begin{pmatrix} -\frac{1}{3} \\ 0 \\ -\frac{2}{3} \end{pmatrix}$$

The matrix f_x corresponding to the constrained variables is

$$f_x = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}$$

The second derivative of F with respect to z,

$$F_{zz} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \\ \\ g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{pmatrix} = -\left(\begin{matrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{matrix} \right)$$

and the derivative of F with respect to λ is

$$F_\lambda = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

We evaluate (2.24) to estimate the stability margin of the constrained system,

$$\begin{aligned} \lambda_{c*} &= \frac{(\mathbf{w}^x f_x \mathbf{v}^x)^2}{2\mathbf{w}F_{zz}(\mathbf{v}, \mathbf{v})\ \mathbf{w}F_\lambda} \\ &= \frac{(\frac{2}{3})^2}{2(\frac{16}{9})2} = \frac{1}{16} \end{aligned}$$

Thus, we expect the bifurcation of the constrained system to occur at $\lambda_* = \frac{1}{16}$. Since $\mathbf{w}^x f_x \mathbf{v}^x$ is positive and the constrained system has only one eigenvalue we expect the constraint to be destabilizing. That is, the eigenvalue of the constrained system should be positive at (y_*, λ_*).

The constrained system differential equation is

$$\dot{y} = g(x_*, y, \lambda) = -x_{1_*} - y^2 - \frac{1}{16} - \lambda = \frac{1}{8} - y^2 - \frac{1}{16} - \lambda = -y^2 + \frac{1}{16} - \lambda \quad (A8.3)$$

The eigenvalue of the constrained system is

$$g_y = -2y \qquad (A8.4)$$

which evaluated at (y_*, λ_*) is

$$g_y|_* = \frac{1}{2}$$

indicating that the constraint is destabilizing as predicted. The bifurcation point of the constrained system is the pair (y_{c*}, λ_{c*}) such that (A8.3) and (A8.4) evaluate to zero. The bifurcation point of the constrained system is thus

$$(y_{c*}, \lambda_{c*}) = (\,0 \quad , \quad \tfrac{1}{16}\,)$$

Again, for this simple system the approximation (2.24) is exact.

Small Power System

Finally, we examine the two bus power system previously analyzed in [18][6][5]. The system consists of two generators, a transformer, and a dynamic load with capacitive support. The parameter values used here are identical to those of [18]. The load model includes a dynamic induction motor model with a constant PQ load in parallel. The combined model for the motor and the PQ load is

$$\begin{aligned} P_l =& P_0 + P_1 + K_{pw}\dot{\delta} + K_{pv}(V + T\dot{V}) \\ Q_l =& Q_0 + Q + K_{qw}\dot{\delta} + K_{qv}V + K_{qv2}V^2 \end{aligned}$$

Q is a parameter which varies with the load reactive power demand. The capacitor is accounted for by adjusting E_0, Y_0 and θ_0 to E_0', Y_0' and θ_0' to give the Thevenin equivalent of the circuit with the capacitor.

The system can be described by the following differential equations

$$\begin{aligned} \dot{\delta}_m =& \omega \\ \dot{\omega} =& \frac{1}{M}(-D\omega + P_m + E_m Y_m V \sin(\delta - \delta_m - \theta_m) + E_m^2 Y_m \sin\theta_m) \\ \dot{\delta} =& \frac{1}{K_{qw}}(-K_{qv2}V^2 - K_{qv}V + Q_l - Q_0 - Q) \\ \dot{V} =& \frac{1}{TK_{qw}K_{pv}}(K_{pw}P_{qv2}V^2 + (K_{pw}K_{qv} - K_{qw}K_{pv})V \\ &+ K_{qw}(P_l - P_0 - P_1) - K_{pw}(Q_l - Q_0 - Q)) \end{aligned}$$

where the real and reactive powers supplied to the load by the network are

$$\begin{aligned} P_l =& -E_0'Y_0'V\sin(\delta + \theta_0') - E_m Y_m V \sin(\delta - \delta_m + \theta_m) \\ &+ (Y_0' \sin\theta_0' + Y_m \sin\theta_m)V^2 \\ Q_l =& E_0'Y_0'V\cos(\delta + \theta_0') + E_m Y_m V \cos(\delta - \delta_m + \theta_m) \\ &- (Y_0' \cos\theta_0' + Y_m \cos\theta_m)V^2 \end{aligned}$$

E_m is the terminal voltage of the generator and E is the internal voltage of the generator. The state vector is

$$z = \begin{pmatrix} \delta_m \\ \omega \\ \delta \\ V \end{pmatrix}$$

and M, D and P_m are the generator inertia, damping and mechanical power respectively. The load parameter values are $K_{pw} = 0.4$, $K_{pv} = 0.3$, $K_{qw} = -0.03$, $K_{qv} = -2.8$, $K_{qv2} = 2.1$, $T = 8.5$, $P_0 = 0.6$, $Q_0 = 1.3$, $P_1 = 0.0$ and the network and generator parameter values are $Y_0 = 20.0$, $\theta_0 = -5.0$, $E_0 = 1.0$, $C = 12.0$, $Y_0' = 8.0$, $\theta_0' = -12.0$, $E_0' = 2.5$, $Y_m = 5.0$, $\theta_m = -5.0$, $E_m = 1.0$, $P_m = 1.0$, $D = 0.12$, $M = 0.3$, $X_s = 0.15$.

We study the effect of eliminating the swing dynamics of the generator on the voltage collapse of this system. The constraint vector is

$$x = \begin{pmatrix} \delta_m \\ \omega \end{pmatrix}$$

We use the numeric and symbolic capabilities of the Mathematica computer algebra package[Wolfram] to find the bifurcation points of both the complete and reduced systems, as well to compute each expression required for calculation of (2.24). The saddle node bifurcation point of the complete system is

$$z_* = \begin{pmatrix} \delta_m \\ \omega \\ \delta \\ V \end{pmatrix} = \begin{pmatrix} 0.3476 \\ 0 \\ 0.1380 \\ 0.9250 \end{pmatrix}$$

at the load of $\lambda_* = Q1_* = 11.4115$. The eigenvalues at the bifurcation are 0, $-0.5729 \pm i2.7386$, and -89.1104. The zero eigenvectors at the bifurcation are

$$\mathbf{w} = (0.1391, 0.3479, 0.0070, -0.9375) \qquad \mathbf{v} = \begin{pmatrix} 0.2418 \\ 0 \\ 0.1050 \\ -1.0300 \end{pmatrix}$$

and the matrix f_x is

$$f_x = \begin{pmatrix} 0 & 1 \\ -15.3010 & -0.4000 \end{pmatrix}$$

Calculation leads to

$$\mathbf{w}^x f_x \mathbf{v}^x = -1.2873$$

and

$$\mathbf{w}F_{zz}(\mathbf{v},\mathbf{v}) = 165.5990, \quad \mathbf{w}F_\lambda = 5.1369$$

Evaluation of (2.24) predicts a change in margin of only 0.0010 between the complete and reduced systems, negligible compared to the magnitude of the load at the bifurcation. The bifurcation point of the reduced system occurs at

$$y_{c*} = \begin{pmatrix} \delta \\ V \end{pmatrix} = \begin{pmatrix} 0.1384 \\ 0.9185 \end{pmatrix}$$

with a loading of $\lambda_{c*} = Q1_{c*} = 11.4121$. The actual change in margin is computed to be 0.0006, not exactly matching the prediction, but also negligible compared to the critical loading. Thus for the bifurcation of this simple test system, the swing dynamics of the generator do not affect the voltage collapse.

Appendix 9. Participation Factors and Voltage Collapse

In [10], Selective Modal Analysis as developed in [27] is applied to voltage collapse, including a case study with a realistic 3700 bus electric power system. Selective Modal Analysis was created for the analysis and reduction of large, linear, time invariant dynamic models used to study dynamic stability in power systems, and hinges upon the interpretation of participation factors, the products of corresponding components of the left and right eigenvectors. The term participation factor was chosen to reflect the fact that these products represent the participation of a state variable in a given mode [27], although this interpretation does not extend well to the case of negative or complex participation factors. Participation factors also represent the sensitivity of eigenvalues to changes in elements of a matrix, as noted in [27], and derived in [15]. In fact, this interpretation of participation factors probably predates both the naming and use of these objects in power systems[14]. In addition, the concept of negative or complex participation factors is then clearly related to the tangent vector to the direction the associated eigenvalue moves in the complex plane for change in the corresponding element of the matrix.

[13] apply modal analysis to a linearization of the voltage stability problem, relating the interpretation of participation factors to classical control systems theory. They demonstrate that participation factors are related to transfer function residues and controllability factors, and illustrate the practicality of this approach with a small test system. In [3], a calculation is presented that describes the change in stability of a dynamic system encountering a constraint. The calculation is similar to (2.26) and can be interpreted in terms of participation factors. In this thesis, participation factors appear in many ways including (2.26) and (2.24). In particular for $x \in \mathbb{R}^1$, the product $\mathbf{w}^x \mathbf{v}^x$ is the participation of the x state variable in the mode associated with the zero eigenvalue. In this thesis we note the importance of the corresponding term $\mathbf{w}^x f_x \mathbf{v}^x$ in determining the stability of the system after encountering the constraint, as well as estimating the change in margin from (2.24).

In the case of power system models that include both algebraic and differential equations, the use of participation factors alone to assess the contribution of a state variable to voltage collapse seems inappropriate. To show this, consider a differential algebraic system described by the equations:

$$
\begin{aligned}
0 &= f(z, \lambda) \\
\dot{y} &= g(z, \lambda) \qquad (A9.1)
\end{aligned}
$$

This system has the same equilibria and solution trajectories as:

$$0 = 2f(z,\lambda)$$

$$\dot{y} = g(z,\lambda) \tag{A9.2}$$

Assume both systems have a zero eigenvalue, then the corresponding right eigenvectors are the same for both the linearization of (A9.1) and (A9.2), but the left eigenvectors are different. Specifically, the x–components of the left eigenvector for (A9.1) are twice that for (A9.2). Thus, the participation factors for the x–components are half those of the second. However, if we are considering approximating the behavior of the system when the x variables are constrained to the equilibrium values determined from the first equations for each system, the reduction of both (A9.1) and (A9.2) should be the same, since solutions of

$$0 = f(z,\lambda)$$

must also be solutions for

$$0 = 2f(z,\lambda)$$

so that in either case we are constraining the same variables to the same values. Note that the calculations for margin (2.24) and ($\mathbf{w}^x f_x \mathbf{v}^x$) give the same result for either system (A9.1) or (A9.2).

Appendix 10. Static Equations

The application of the results of this thesis to static models requires further comment. If a constraint on a dynamic model limits a state variable to a particular value, it is sensible to eliminate the equation describing that variable's time derivative from the model. However, if the model is static, there is no particular reason to eliminate one equation rather than another. Note that for some static problems, there may in fact be a connection between each equation and each variable. For example, assume that (1.2) describes the static conditions for equilibrium of the DC load flow problem of a power system, and it is desired to know the effect of holding the voltage at one bus constant. In this case, it would seem logical to dispose of the equation representing load balance at the bus for which the voltage is to be constrained. In general, however, we can not assume that there is a one to one correspondence between the variables and the equations of a static model as there is for a dynamic model.

For static models, the signs and magnitudes of non–zero eigenvalues do not have any stability interpretation, and hence, calculation concerning eigenvalue sensitivity is irrelevant. Without dynamics, there is no justification to prefer an eigenvalue with negative real part to one with positive real part. Nevertheless, the mathematics is still valid, only the interpretations of the result changes. If (1.2) reflects purely static conditions of equilibrium, then (2.23) and (2.24) are estimates of the distance to bifurcation of a new set of algebraic equations, based upon the elimination of one or more of the equations and variables. When solutions of (1.2) represent equilibria of a dynamic system (1.1), then we may extend the interpretation of (2.23) and (2.24) to be related to the system encountering a constraint. In addition, for these dynamic models, the sign of $\mathbf{w}^x f_x \mathbf{v}^x$ determines the sign of the real eigenvalue closest to the origin after encountering the constraint, and the entire system is stable if the eigenvalues of $g_y|_{(y_*,\lambda_*,\theta_*)}$ have negative real parts.

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