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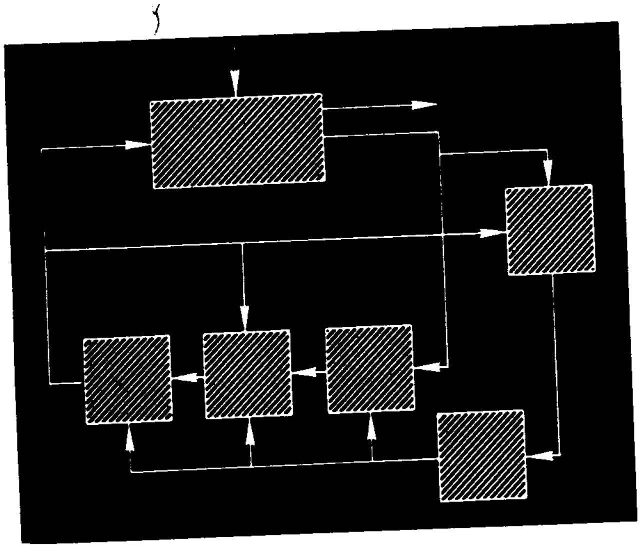

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Towards a theory of voltage collapse in electric power systems

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Abstract: Several recent major power system blackouts are characterised by a progressive decline in voltage magnitude at the system buses. These events are termed 'voltage collapses'. The mechanisms of voltage collapse are not well defined and the dynamics of the process are not well understood. In this paper, we describe the loss of stability when a stable equilibrium point disappears in a saddle node bifurcation and present a simple model of the system dynamics after the bifurcation. The results apply generally to any generic one parameter dynamical system. Then we use these results to propose a model for voltage collapse in power systems. The model gives an explicit mechanism for the dynamics of voltage collapse. We illustrate the model by constructing a simple power system model and simulating a voltage collapse.

Keywords: Voltage collapse; power systems; bifurcation; saddle node; center manifold; dynamics.

1. Introduction

Most of the major electric power system breakdowns in recent years have been caused by the dynamic response of the system to disturbances. Moreover, economic and environmental pressures are causing power systems to be operated ever closer to their limits of stability. Thus dynamic security assessment of power systems is becoming increasingly important.

One type of system instability which occurs when the system is heavily loaded is voltage collapse. Two recent examples of voltage collapse leading to system blackout occurred in France, December 1978 and in Belgium, August 1981 [2,4]. Both events were characterised by a slow decline in voltage magnitude at buses over a period of minutes and hours followed by a sharp decrease in voltage magnitude.

An international effort to clarify the mechanisms of voltage collapse has yielded many approaches to the problem but no consensus on the mechanisms involved. A major issue is whether voltage collapse is a static or a dynamic event. Reviews of these approaches may be found in [6,14,10]. In particular, few authors have attempted to describe the dynamics of voltage collapse. Liu in [11] presented a dynamical description of voltage collapse of a nonlinear on-line tap-changer model based on characterising the voltage stability region in terms of the tap-changer setting. This model is extended in [12] to include an impedance-type load model and a decoupled reactive load flow equation. Medanić et al. [13] investigate the voltage stability of discrete models of multiple tap-changers in a power network. In [20], Thomas and Tiranuchit present a mechanism describing voltage collapse by taking load dynamics into account and showing its effect on the stability region. These mechanisms are promising in their description of dynamics of voltage collapse but the qualitative features of voltage collapse are not explained.

In this paper, we suggest a dynamic mechanism for power system voltage collapse with voltage magnitudes decreasing slowly at first and then decreasing rapidly. This mechanism arises from a description [17] of a generic saddle node bifurcation. The essential point is that at such a bifurcation, the system state will leave the bifurcating equilibrium point and move along a particular trajectory. The movement along the trajectory is slow at first and then more rapid. If bus voltages decrease along this trajectory then we identify the movement along the trajectory with voltage collapse. Thus we suggest that voltage collapse be explained as a dynamic consequence of the bifurcation.

After briefly reviewing some essential dynamical systems concepts in Section 2, we describe our modelling assumptions and present a simple model of the dynamics near a generic saddle node bifurcation in Section 3. Section 4 supplies the precise description and conditions needed from [17]. Section 5 discusses the form of power system models to which the theory naturally applies and the relationship to previous work. Section 6 constructs a simple example of a suitable power system model and illustrates how the voltage collapse theory applies to this example. Section 7 discusses the status of the voltage collapse theory and presents the conclusions.

2. Preliminaries

We briefly review some notions from nonlinear dynamical systems theory which are needed in the sequel. The details may be found in [9].

Consider a nonlinear dynamical system described by

$$\dot{x} = f(x) \tag{2.1}$$

where $f: \mathrm{R}^n \to \mathrm{R}^n$ is assumed to satisfy the conditions for existence and uniqueness of solutions. A point x_0 is called an *equilibrium point* of (2.1) if $f(x_0) = 0$. We say that the equilibrium point x_0 is *hyperbolic* if the Jacobian matrix

$$\frac{\partial f}{\partial x}(x_0) \tag{2.2}$$

has no eigenvalues with zero real part. The equilibrium point x_0 is said to be *simple* if the determinant of the Jacobian matrix (2.2) is nonzero. The *type* of the hyperbolic equilibrium point x_0 is defined to be the number of eigenvalues of (2.2) with positive real part. For example, x_0 is a type one equilibrium point if (2.2) has exactly one eigenvalue with positive real part and x_0 is a type zero equilibrium point if x_0 is stable.

The *unstable manifold* $W^{\mathrm{u}}(x_0)$ (*stable manifold* $W^{\mathrm{s}}(x_0)$) of an equilibrium point x_0 is the manifold in the state space from which trajectories converge to x_0 as $t \to -\infty$ ($t \to \infty$) and which is tangent at x_0 to the subspace spanned by the (generalised) eigenvectors associated with eigenvalues with positive (negative) real parts. (Some authors call the manifold defined above the strong unstable (stable) manifold when x_0 is not hyperbolic.) If x_0 is hyperbolic, the dimension of $W^{\mathrm{u}}(x_0)$ is equal to the type of x_0. For a nonhyperbolic equilibrium point x_0, there exists another invariant set, called the *center manifold* $W^{\mathrm{c}}(x_0)$, which is tangent to the subspace spanned by the (generalised) eigenvectors associated with the eigenvalues of (2.2) on the imaginary axis. The stable and unstable manifolds are unique, but the center manifold may be nonunique.

3. Overview of theory

This section states our modelling assumptions and the relevant conclusions from Sotomayor's theory of generic bifurcations [17]. Further modelling assumptions are made in order to obtain a simplified description of the dynamics near a generic saddle node bifurcation. The details and precise statement of Sotomayor's theory are postponed to Section 4.

Suppose a system is defined by the differential equation

$$\dot{x} = X_\lambda(x) \tag{3.1}$$

where x is an n dimensional state vector and λ is a time varying parameter. We approximate system (3.1) by assuming:

Assumption 1. λ varies quasistatically.

That is, we assume that λ varies slowly enough that system (3.1) with time varying λ is well approximated by keeping λ constant while the dynamics of system (3.1) act. For example, if system (3.1) has a stable equilibrium point x_0^λ and the system state x is initially near x_0^λ then the dynamics will make x track x_0^λ as λ and x_0^λ change slowly.

We further assume:

Assumption 2. System (3.1) is in the generic set of systems Γ_1.

Γ_1 is a generic set of systems described by Sotomayor; Γ_1 consists of the systems of the form (3.1) which, for each λ, have all simple equilibria except that it is possible for one of the equilibria to be a nondegenerate saddle node equilibrium. A more precise definition of Γ_1 is given in Section 4. Assumption 2 is important and desirable because it implies that systems in Γ_1 represent typical behaviour of physical systems (e.g., we expect the bifurcations to be observed in applications) and that they are robust to small modelling variations.

We now state the conclusions from Sotomayor's theory and describe the resulting dynamical structure near the bifurcation.

Fact 1. Suppose system (3.1) satisfies Assumption 2. Then the only way in which a stable equilibrium point x_0^λ can disappear is by coalescing with a type one equilibrium point x_1^λ in a saddle–node bifurcation. Just before the bifurcation, x_1^λ is on the stability boundary of x_0^λ and x_1^λ is the closest unstable equilibrium point to x_0^λ.

There are two typical ways in which a stable equilibrium point can loss stability; either it disappears as stated in Fact 1 or it persists but becomes unstable by interacting with a limit cycle in a Hopf bifurcation. Fact 1 does not exclude Hopf bifurcations but we note that many power system models do not admit limit cycles and hence cannot have Hopf bifurcations [1,5]. In any case, we focus here on the disappearance of stable equilibrium points and exploit Fact 1 to give the following picture of the disappearance.

While an equilibrium point x_0^λ is stable, it lies in the interior of its stability region. x_0^λ can only disappear by bifurcating with an equilibrium point x_1^λ on its stability boundary. Fact 1 states that x_1^λ must be type one, that is, its unstable manifold $W^u(x_1^\lambda)$ is one dimensional. $W^u(x_1^\lambda)$ may be decomposed as

$$W^u(x_1^\lambda) = W_-^u \cup \{x_1^\lambda\} \cup W_+^u.$$

W_-^u lies inside the stability region of x_0^λ and joins x_0^λ to x_1^λ while W_+^u lies outside the stability region of x_0^λ (see Figure 1).

At the bifurcation, $\lambda = \lambda_*$ and x_0^λ and x_1^λ coalesce to form the equilibrium point $x_* = x_0^{\lambda_*} = x_1^{\lambda_*}$. The Jacobian at x_* has a zero eigenvalue with an eigenvector w in the direction in which x_0^λ and x_1^λ coalesced. The other $n-1$ eigenvalues of the Jacobian of x_* remain negative. Therefore x_* has a one dimensional center manifold W^c and an $n-1$ dimensional stable manifold $W^s(x_*)$. W^c may be decomposed as

$$W^c = W_-^c \cup \{x_*\} \cup W_+^c$$

and w is tangent to W^c at x_*. The vector field at

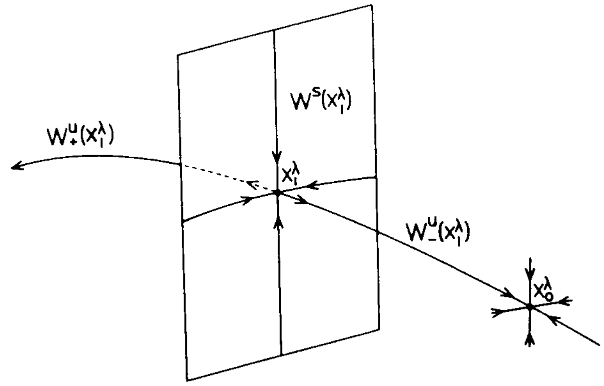

Fig. 1. Just before bifurcation.

x_* has one sided stability along W^c; x_* is stable along W_-^c and unstable along W_+^c (see Figure 2). W_+^c is a unique system trajectory. Note that W_+^u becomes W_+^c as the bifurcation occurs.

Now we consider how to make further modelling assumptions to simplify the dynamics near the bifurcation. While the stable equilibrium point persists, Assumption 1 implies that the system state x tracks the stable equilibrium point x_0^λ. At the bifurcation, we again make Assumption 1 to idealise and simplify the system behaviour. We consider how the dynamics of system (3.1) act on x when it is initially at x_* and λ is fixed at the bifurcation value λ_*. The stable manifold $W^s(x_*)$ divides a neighborhood of x_* into a region containing W_-^c in which trajectories converge to x_* and a region containing W_+^c in which trajectories diverge from x_*. The system state x cannot remain at the equilibrium point x_* because x_* is unstable; any small perturbation of x into the region containing W_+^c will result in x diverging from x_*. We make the following assumption about

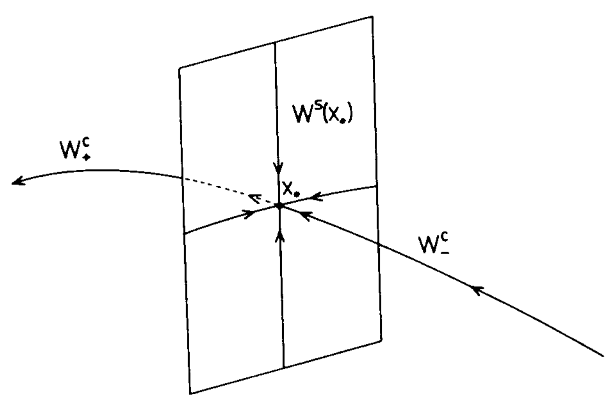

Fig. 2. At bifurcation.

the perturbation to simplify and approximate the dynamics as x diverges from x_*.

Assumption 3. Suppose system (3.1) satisfies Assumptions 1 and 2 and has a saddle node bifurcation and the system state x is at the bifurcating equilibrium point x_*. Then x leaves the unstable equilibrium point x_* by being perturbed to a point on W_+^c very close to x_*.

Thus at the bifurcation x is slightly perturbed to lie on W_+^c and then the system dynamics move x along W_+^c. The initial movement along W_+^c is slow since near x_* the dynamics are dominated by the zero eigenvalue of the linearised dynamics along W_+^c at x_*. When the system state is no longer close to x_*, we expect the movement along W_+^c to be rapid.

We discuss why Assumption 3 is a sensible simplification of the dynamics. Suppose the perturbation moves x from x_* to the region containing W_+^c on one side of the stable manifold of x_* but not necessarily on W_+^c. All trajectories starting from this region approach W_+^c exponentially fast since the $n-1$ nonzero eigenvalues of the linearisation at x_* are negative and the initial movement along W_+^c is slow. Therefore the perturbed trajectories are locally well approximated by corresponding trajectories on W_+^c (this can be proved if an additional generic assumption is made; see Appendix). Approximating the perturbed trajectories by the corresponding trajectories on W_+^c is equivalent to restricting the perturbations using Assumption 3. Another alternative would be to consider what happens as the perturbation becomes infinitesimally small. The system state x would indeed move along W_+^c, but a trajectory on W_+^c starting infinitesimally close to x_* would take infinite time to move a finite distance along W_+^c. Therefore we prefer to consider small, finite perturbations subject to Assumption 3. A feel for the typical dynamics at a saddle note bifurcation may be obtained by inspecting the dynamics in the xy plane of $\dot{x} = ax^2$, $\dot{y} = -by$ with a and b positive constants. In this case W_+^c is the positive x axis.

Thus given Assumptions 1, 2 and 3, we obtain the central result of the paper:

Fact 2. Suppose system (3.1) satisfies Assumptions 1, 2 and 3. Then at the saddle node bifurcation of a stable equilibrium point, the center manifold is one dimensional and the unstable part of the center manifold W_+^c is a unique system trajectory. At the bifurcation the equilibrium point x_* is unstable and the system state will move along W_+^c.

4. Details of theory

Sotomayor's paper [17] precisely describes a set of one parameter vector fields which include saddle node bifurcations and proves that this set is generic in the space of all one parameter vector fields. Now we extract from Sotomayor's paper the special case of interest to us:

Let M be a compact C^∞ manifold [1] and let Φ be the set of all C^r tangent vector fields on M with the C^r topology, where $r \geq 2$. Fix an interval $I = [\lambda_1, \lambda_2]$ on the real line and consider one parameter families of vector fields

$$\begin{aligned} \xi\colon\ & I \to \Phi \\ & \lambda \mapsto X_\lambda; \end{aligned}$$

each such ξ defines a curve of vector fields in Φ. Each map ξ has an associated map

$$\begin{aligned} \hat{\xi}\colon\ & I \times M \to TM \\ & (\lambda, x) \mapsto X_\lambda(x). \end{aligned}$$

Let Γ be the set of maps ξ: $I \to \Phi$ for which the corresponding map $\hat{\xi}$ is C^r. Give Γ the topology such that ξ, η are close if $\hat{\xi}$, $\hat{\eta}$ are C^r close.

A saddle node equilibrium point x_0^λ of system (3.1) has a Jacobian at x_0^λ with a single zero eigenvalue and satisfies two transversality conditions. The first transversality condition requires a nonzero quadratic term in the flow reduced to a center manifold passing through x_0^λ. The second transversality condition requires the curve $\lambda \mapsto X_\lambda$ in Φ to intersect the hypersurface of vector fields with a saddle node near x_0^λ transversally at X_{λ_*}. These transversality conditions are explained in detail in [17] and [9].

The structure at a saddle node bifurcation is as follows: For values of λ near to the bifurcation value λ_* and on one side of λ_*, say $\lambda < \lambda_*$, there are two hyperbolic equilibrium points x_0^λ, x_1^λ near

[1] To satisfy the compactness condition on M it is sufficient to find a compact positively invariant subset of state space. We expect to find such a subset for power system models as long as damping terms are not neglected.

x_*. The types of x_0^λ and x_1^λ differ by one. The stable manifold of x_0^λ, $W^s(x_0^\lambda)$ and the unstable manifold of x_1^λ, $W^u(x_1^\lambda)$ intersect along a one dimensional manifold with endpoints x_0^λ and x_1^λ. For values of λ with $\lambda > \lambda_*$, there are no equilibrium points nearby.

Let Γ_1 be the set of C^r maps in Γ such that

(A) For each $\lambda \in (\lambda_1, \lambda_2)$, X_λ has simple equilibrium points except for at most one saddle node equilibrium point satisfying the two transversality conditions.

(B) X_{λ_1} and X_{λ_2} have only simple equilibrium points.

Sotomayor's theorem is that Γ_1 is generic in Γ in the sense that Γ_1 is open and dense in Γ. We are particularly interested in the bifurcations occurring in Γ_1 in which x_0^λ is stable. In these bifurcations, x_1^λ is type one and at $\lambda = \lambda_*$, x_* has a one dimensional center manifold

$$W^c = W_-^c \cup \{x_*\} \cup W_+^c.$$

W_+^c is a trajectory and a unique one dimensional C^r manifold with boundary x_*. W_-^c is also a trajectory but is not unique. Just before the bifurcation, $W^s(x_0^\lambda)$ and $W^u(x_1^\lambda)$ intersect along W_-^u. Hence W_-^u is contained in $W^s(x_0^\lambda)$, the stability region of x_0^λ, and x_1^λ lies on the stability boundary of x_0^λ. Fact 1 follows.

5. Application to power systems

The theory above applies naturally to the following class of power system models.

Let the system state x include bus angles δ, bus angular velocities ω and bus voltage magnitudes V. Suppose that the power system is modelled as a system of the form (3.1) depending on a single parameter λ, where λ is a slowly varying function of time with values in the interval $I = [\lambda_1, \lambda_2]$. λ might typically be a reactive power demand.

It would be desirable to develop a voltage collapse theory for the case of a vector of parameters λ so that saddle node bifurcations would still arise generically when several power system parameters are freely varying. However we note that the single parameter theory is probably sufficient to illustrate voltage collapse in particular power system models. For example, Tamura et al. [19] give examples of saddle node bifurcations associated with voltage collapse due to variation of a single reactive power injection parameter.

Now we make Assumptions 1 and 2. In particular, we assume that the power system model (3.1) is in the generic set Γ_1 and that variations in λ are slow enough to be modelled as quasistatic variations. We also make Assumption 3 to simplify the dynamics near bifurcation. The consequences are that stable equilibrium points of (3.1) can only disappear by saddle node bifurcation with a type one unstable equilibrium point on its stability boundary (Fact 1) and that at such a bifurcation the system state will move along the trajectory W_+^c (Fact 2).

The dynamic consequences of the bifurcation are determined by the position of W_+^c in state space. For example, W_+^c might join x_* to a stable equilibrium point $x_2^{\lambda_*}$ and voltages V might be approximately constant along W_+^c but the angle δ might vary significantly along W_+^c. Then the consequence of the bifurcation is pole slip until $x_2^{\lambda_*}$ is reached. The system would subsequently track x_2^λ.

Another possibility is that W_+^c is positioned in state space so that components of V decrease along W_+^c. We propose this movement along W_+^c as a model for voltage collapse:

Model 1. Suppose power system model (3.1) has a saddle node bifurcation and W_+^c is positioned in state space so that some components of V decrease along W_+^c. Then the movement of the state vector along W_+^c starting near x_* is a model for voltage collapse.

The initial direction of W_+^c is along the eigenvector w corresponding to the zero eigenvalue of the Jacobian at x_*. This allows the case of voltage collapse and pole slip, or some combination of the two, to be distinguished, at least in the initial movement along W_+^c. Viewing voltage collapse and pole slip as different cases of the same phenomenon is due to Kwatny et al. [10].

Model 1 predicts that the initial voltage decrease is slow because the initial movement along W_+^c is slow. We also expect the subsequent movement along W_+^c to be rapid, causing a rapid decrease in voltage. This prediction agrees qualitatively with features of observed voltage collapses [2,4].

However another mechanism may also contribute to the observed slow initial voltage decrease.

Consider the system before a saddle node bifurcation. Under Assumption 1, the system can be modelled by the system state tracking x_0^λ as λ varies. The slow variation in λ will generally cause the corresponding movement of x_0^λ to be slow. Thus voltage magnitudes may well decrease slowly before bifurcation as well as at bifurcation. (We expect voltages to decrease before a saddle node bifurcation leading to voltage collapse according to Model 1 because x_0^λ approaches x_* along the direction of the eigenvector w and since w is tangent to W_+^c at x_*, w must point in a direction in which some components of V decrease.)

Kwatny et al. [10] associate voltage collapse with a bifurcation at which the load voltages are infinitely sensitive to parameter variations. For a similar sensitivity viewpoint see [7]. Model 1 allows this association to be viewed differently. Saddle node bifurcations in which the load voltage becomes infinitely sensitive to parameter variations as the bifurcation is approaches are exactly those which have an eigenvector w with a zero eigenvalue and nonzero components in the voltage direction. Thus we do expect infinite voltage sensitivity at a bifurcation which is associated with a voltage collapse but we choose to explain the voltage collapse by the subsequent movement along W_+^c.

Several issues raised in Tamura et al. [19] can be clarified if the power systems in [19] can be modelled as one parameter generic models of the form (3.1). The assumption in [19] that in a bifurcating pair of equilibrium points, one is stable and the other unstable is verified. Moreover, the unstable equilibrium point is type one (Fact 1). We agree that voltage sensitivity to parameter variations can be defined for both stable and unstable equilibrium points, but argue that such sensitivities are only meaningful at stable equilibrium points because any solution near an unstable equilibrium point will leave that equilibrium point.

One goal of our research is to construct convincing power system models of the form (3.1), locate a saddle node bifurcation, and study the ensuing voltage collapses in order to test the voltage collapse model 1 on specific examples. The example presented in Section 6 is simple, but an important step towards this goal. Before presenting this example, we discuss the problem of constructing a suitable power system model.

There are few power system models of the form (3.1) because little is known about the dynamics of load voltage magnitudes V_L [20]. Most power system models with varying V_L include algebraic equations as well as differential equations. Typically, other state variables are specified by differential equations and V_L is specified by solving algebraic equations. The algebraic equations are presumably idealisations of some unmodelled dynamics which normally tend to act so that the algebraic equations are satisfied. Thus one problem is to develop dynamics for V_L which somehow generalise the algebraic equations. DeMarco and Bergen pursue this in [8] using singular perturbation ideas [16]. They start with a structure preserving model [3,15] and add a term $\varepsilon \dot{V}_L$ to the load reactive power balance equation to obtain dynamics for V_L. (ε is a small positive parameter.) This does indeed yield a power system model of the form (3.1). However, we are unsure how to choose a value of ε. In the singular perturbation limit $\varepsilon \to 0+$, the speed of the dynamics in most of the state space becomes infinite and we would expect the slow initial movement along W_+^c to be

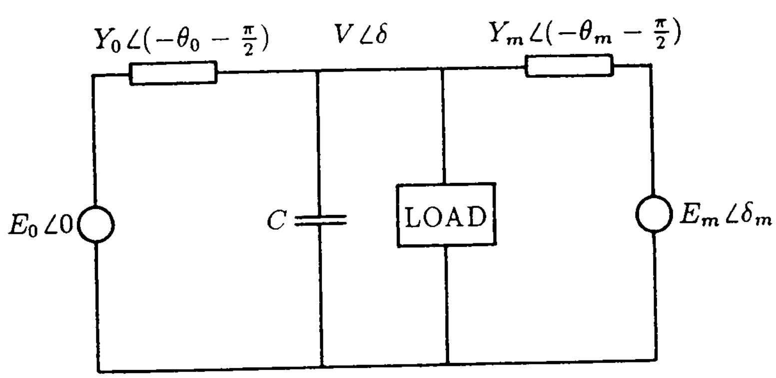

Fig. 3. Power system example.

destroyed. On the other hand, we do not know how to physically justify the choice of a larger value of ε.

6. Example

This section summarises an example from [21] to illustrate how voltage collapse model 1 applies to the power system model shown in Figure 3. One generator is a slack bus and the other generator has constant voltage magnitude E_m and angle dynamics given by the swing equation

$$M\ddot{\delta}_m = -d_m\omega + P_m + E_m V Y_m \sin(\delta - \delta_m - \theta_m) + E_m^2 Y_m \sin\theta_m, \tag{6.1}$$

where M, d_m and P_m are the generator inertia, damping and mechanical power respectively.

The load model includes a dynamic induction motor based on a model due to Walve [22] and a constant PQ load in parallel. The induction motor model specifies the real and reactive power demands P and Q of the motor in terms of load voltage V and frequency $\dot{\delta}$. The combined model for the motor and the PQ load is

$$P = P_0 + P_1 + K_{p\omega}\dot{\delta} + K_{pv}(V + T\dot{V}), \tag{6.2a}$$

$$Q = Q_0 + Q_1 + K_{q\omega}\dot{\delta} + K_{qv}V + K_{qv2}V^2, \tag{6.2b}$$

where P_0, Q_0 are the constant real and reactive powers of the motor and P_1, Q_1 represent the PQ load.

Q_1 is chosen as the system parameter so that increasing Q_1 corresponds to increasing the load reactive power demand. The load also includes a fixed capacitor C to raise the voltage up to near 1.0 per unit. Instead of including the capacitor in the circuit, it is convenient to account for the capacitor by adjusting E_0, Y_0 and θ_0 to give the Thévenin equivalent of the circuit with the capacitor. The adjusted values are denoted by E_0', Y_0' and θ_0'.

The real and reactive powers supplied to the load by the network are

$$\begin{aligned} P(\delta, V) = &-E_0'Y_0'V \sin(\delta + \theta_0') \\ &- E_m Y_m V \sin(\delta - \delta_m + \theta_m) \\ &+ (Y_0' \sin\theta_0' + Y_m \sin\theta_m)V^2, \end{aligned}$$

$$\begin{aligned} Q(\delta, V) = &\,E_0'Y_0'V \cos(\delta + \theta_0') \\ &+ E_m Y_m V \cos(\delta - \delta_m + \theta_m) \\ &- (Y_0' \cos\theta_0' + Y_m \cos\theta_m)V^2. \end{aligned}$$

Putting equation (6.1) in state variable form and rearranging equations (6.2) so that $\dot{\delta}$ and $\dot{V}$ appear as the left hand side we obtain the system differential equations in the form of equation (3.1).

$$\dot{\delta}_m = \omega, \tag{6.3a}$$

$$M\dot{\omega} = -d_m\omega + P_m + E_m Y_m V \sin(\delta - \delta_m - \theta_m) + E_m^2 Y_m \sin\theta_m, \tag{6.3b}$$

$$K_{q\omega}\dot{\delta} = -K_{qv2}V^2 - K_{qv}V + Q(\delta, V) - Q_0 - Q_1, \tag{6.3c}$$

$$\begin{aligned} TK_{q\omega}K_{pv}\dot{V} = &\,K_{p\omega}K_{qv2}V^2 \\ &+ (K_{p\omega}K_{qv} - K_{q\omega}K_{pv})V \\ &+ K_{q\omega}(P(\delta, V) - P_0 - P_1) \\ &- K_{p\omega}(Q(\delta, V) - Q_0 - Q_1). \end{aligned} \tag{6.3d}$$

Thus the dynamic load model (6.2) solves the problem of obtaining differential equations of the form (3.1) for this power system model.

We find a compact, positively invariant subset C of the state space $S^1 \times R \times S^1 \times R$ of (6.3) as required in Section 4. Let C be the compact set

$$S^1 \times [-\omega_1, \omega_1] \times S^1 \times [-V_1, V_1],$$

where ω_1 and V_1 are chosen large enough that the vector field (6.3) points inwards on the boundary of C so that C is positively invariant. This is possible since for large ω the second equation of (6.3) is dominated by $\dot{\omega} = -M^{-1}d_m\omega$ and the coefficient $-M^{-1}d_m$ is negative and for large V the fourth equation of (6.3) is dominated by

$$\dot{V} = (TK_{qw}K_{pv})^{-1}K_{p\omega}K_{qv2}V^2$$

and the coefficient of V^2 is negative. Then for large ω_1 and V_1, the vector field points inwards on the hyperplanes $\omega = \pm\omega_1$ and $V = \pm V_1$ and regions of these hyperplanes form the boundary of C.

A saddle node bifurcation was found by solving equations (6.3) with left hand sides zero and the determinant of the Jacobian of these equations set to zero for the variables δ_m, ω, δ, V, Q_1. The load parameter values were

$K_{p\omega}=0.4$, $K_{pv}=0.3$,
$K_{q\omega}=-0.03$, $K_{qv}=-2.8$, $K_{qv2}=2.1$,
$T=8.5$, $P_0=0.6$, $Q_0=1.3$, $P_1=0.0$

and the network and generator parameter values were

$Y_0=20.0$, $\theta_0=-5.0$, $E_0=1.0$, $C=12.0$,
$Y_0'=8.0$, $\theta_0'=-12.0$, $E_0'=2.5$,
$Y_m=5.0$, $\theta_m=-5.0$, $E_m=1.0$,
$P_m=1.0$, $d_m=0.05$, $M=0.3$.

All values are in per unit except for angles, which are in degrees. The parameters were adjusted to produce an example of saddle node bifurcation with V near 1 per unit and with small (<20 degrees) line angles.

The bifurcating equilibrium was

$$x_* = (\delta_m^*, \omega^*, \delta^*, V^*) = (0.348, 0.0, 0.138, 0.925)$$

and the bifurcation value of the parameter was $Q_1^* = 11.41$. (All values are in per unit except for angles, which are in radians.) The eigenvector with zero eigenvalue was

$$w = (0.23, 0.0, 0.099, -0.97).$$

The relatively large negative component of voltage in w shows that W_+^c is oriented so that the voltage will decrease at the bifurcation, at least initially. To confirm this and to determine the character of the collapse along W_+^c in this case, equations (6.3) were numerically integrated starting from an initial condition displaced by 0.01 from x_* in the direction of w (Assumption 3). Q_1 was held fixed at Q_1^* throughout the integration (Assumption 1). Figure 4 shows the resulting voltage profile. The load voltage decrease is initially slow and then rapid. Note that in the later stages of the collapse, the low voltage would cause protection devices to trip, thus changing the assumed power system model.

Let I be a short closed interval containing Q_1^*. Then the curve of systems obtained by mapping $Q_1 \in I$ to equations (6.3) with Q_1 as a parameter is in the generic set of systems Γ_1 and satisfies Assumption 2. (The saddle node bifurcation satisfies the two transversality conditions mentioned in Section 4.) In particular, since Γ_1 is open, the saddle node bifurcation is robust to small perturbations of equations (6.3).

Model 1 can also be applied to the power system models used to explain voltage collapse in Tiranuchit, Thomas and Liu [11,20]. Their voltage collapse models can be viewed as saddle node bifurcations in one dimensional state spaces and

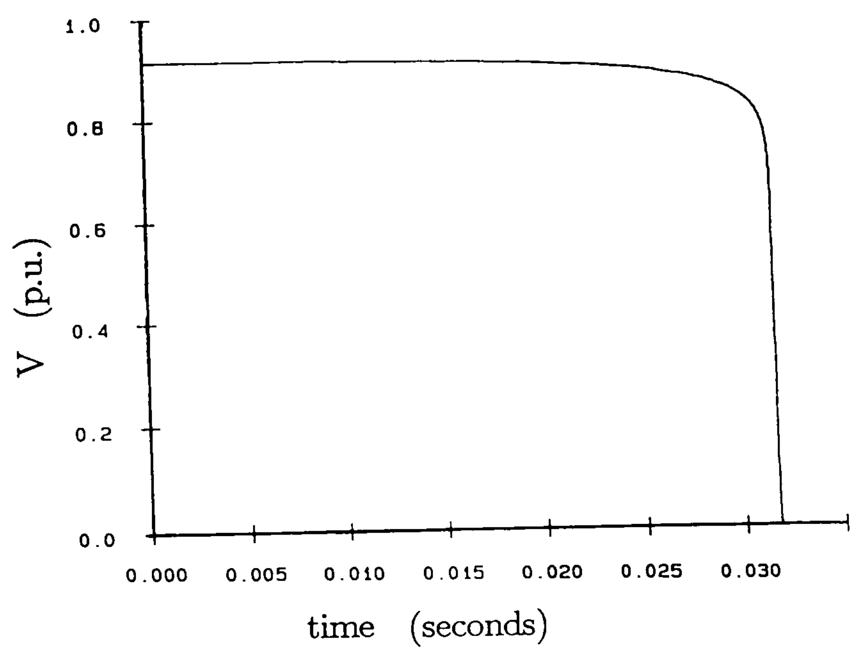

Fig. 4. Load voltage magnitude during voltage collapse.

are consistent with Model 1 if Assumptions 1, 2 and 3 are made. (Note that [11,20] consider discrete parameter changes while we consider a slowly varying parameter (Assumption 1).) At a saddle node bifurcation in a one dimensional state space, the center manifold is the entire state space and movement along W_+^c which is initially slow and then more rapid will follow.

7. Discussion and conclusions

Suppose we are given a power system model of the form (3.1) and we attempt to better represent the system behaviour by also modelling perturbations of both the state vector x and the parameter λ. When the power system model is approximately at a bifurcation point (x_*, λ_*) the perturbations may cause (x, λ) to vary in a complicated manner near (x_*, λ_*) before x moves approximately along W_+^c. The voltage collapse model 1 is an idealisation of this complicated situation; our intent is to make sensible modelling assumptions in order to obtain the simplest possible model which captures an essential mechanism of voltage collapse.

This paper considers a generic power system model tracking a slowly varying stable equilibrium point. The power system is modelled as a set of differential equations with a single, slowly moving parameter. This generic model can typically lose stability by a saddle node bifurcation and at the bifurcation, the dynamics can be modelled by the movement of the system state along the particular trajectory W_+^c. We note that this simplified model of the dynamics after bifurcation of a stable equilibrium point applies to any generic system of differential equations with a single, slowly moving parameter. We propose the movement along W_+^c as a model for voltage collapse. This model for voltage collapse is static in that the parameter is assumed to be fixed during the collapse but dynamic in that the system is not at an equilibrium point during the collapse. The movement along W_+^c is initially slow and is nonlinear, giving a qualitative explanation of the initially slow and subsequently rapid voltage decrease observed in voltage collapse. We note that since the voltage collapse model predicts movement along a particular trajectory, the voltage collapse predicted by any suitable power system model may be calculated by numerical integration. Although the voltage collapse model applies to a very general class of power system models, the dynamics of load voltages need to be modelled to construct these power system models. We show by an example that a suitable power system model can be constructed in this way and demonstrate numerically a voltage collapse with an initially slow and subsequently rapid voltage decrease. Other mechanisms may also cause a slow decrease in voltage before the bifurcation. This simple example is an important step towards demonstrating the validity of the voltage collapse model. We feel that the model is a strong candidate for explaining voltage collapse because of previous work associating voltage collapse with bifurcation, the generic nature of the model, and its qualitative prediction of features observed in voltage collapses.

Appendix

Suppose system (3.1) is C^∞ smooth and let x_* be an equilibrium point of system (3.1) whose Jacobian has a single zero eigenvalue and $n-1$ eigenvalues with negative real parts. We diffeomorphically change coordinates to demonstrate that trajectories in a neighborhood of x_* starting near W_+^c approach W_+^c exponentially fast. If the nonzero eigenvalues of the Jacobian satisfy a nonresonance condition [17,18], then there is a C^1 change of coordinates reducing system (3.1) in a neighborhood N of x_* to the form

$$\dot{x} = ax^2 + o(x^2), \tag{A1a}$$

$$y = A(x)y, \tag{A1b}$$

where $y \in \mathbb{R}^{n-1}$ and $A(0)$ has eigenvalues with negative real parts. The nonresonance condition on the eigenvalues of $A(0)$ is generically satisfied [17,18]. In the new coordinates, x_* is the origin and W_+^c is the positive x axis (we choose $a>0$). If necessary, reduce the size of N so that the eigenvalues of $A(x)$ have negative real parts for all (x, y) in N. Choose $\mu > 0$ so that $\mu \le |\lambda(x)|$ where $\lambda(x)$ is the eigenvalue of $A(x)$ with the smallest modulus. If $(x_0, y_0) \in N$ and $x_0 > 0$, then the distance between the trajectories through (x_0, y_0) and the corresponding trajectory through $(x_0, 0)$ is $|y(t)|$ while the trajectories remain in N. But $|y(t)| \le e^{-\mu t}|y(0)|$. Thus trajectories in $N \cap \{x > 0\}$ are exponentially well approximated by corresponding trajectories in W_+^c. Since the coordinate change was C^1, the same conclusion holds for the trajectories in the original coordinates.

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