How many occurrences of rare blackout events are needed to estimate event probability?

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***Abstract*—We give simple conditions on the number of occurrences needed for statistically valid conclusions about simulated or observed rare events associated with blackouts. For example, 11 occurrences are needed to estimate the probability of a rare event within a factor of 2 with 95% confidence.**

***Index Terms*—power system reliability, statistics, simulation.**

I. Introduction

Series of blackouts can be observed in real power systems or simulated in power system models, and can be regarded as samples of blackouts. In either case, the most important and large blackouts involve rare events [1], [2], [3]. Then it is necessary to consider how many occurrences are needed for statistically valid estimates of the rare event probabilities.[1] The required number of occurrences of rare events has a large effect on practicality, since it governs how long real data must be observed or how many simulation runs are needed.

Consider blackouts in WSCC reported to NERC in 1984. There were 14 blackouts bigger than 100 MW, corresponding to an estimated daily probability 14/364=0.04. There were 5 blackouts bigger than 1000 MW, corresponding to an estimated daily probability 5/364=0.01. This letter will show that the first probability has statistical significance, whereas the second probability does not and requires more observations. Another example of a rare event associated with a blackouts is "a particular sequence of three lines trips during a blackout."... we would want to know how many times this event should occur in order to reliably estimate its probability.

This letter generally raises the rather neglected issue of statistical validity for conclusions about rare blackout events and explains some simple criteria that are easy to apply. The results are given in section II and then derived in section III. We hope that these straightforward results will help to sharpen methods and conclusions for analyzing blackout data.

II. Results

Suppose there are n sample blackouts observed or simulated. The rare event of interest (for example, a blackout of size more than 1000 MW) occurs h times in the n sample blackouts. Then the probability p of the event is estimated as

$$\hat{p} = \frac{h}{n}.$$

We assume that the samples are statistically independent. How many samples n are needed for the estimate $\hat{p}$ to be likely to be a sufficiently accurate answer? Alternatively, given the number of samples n, how small are the probabilities that are likely to be estimated sufficiently accurately? It turns out that the answers can be expressed in terms of the minimum number of occurrences h. The required minimum number of occurrences h depends on the required accuracy and the confidence level. Our accuracy requirement chooses a constant a (such as $a = 2$) and requires that $\hat{p}$ be within a factor a of p:

$$p/a \le \hat{p} \le ap. \tag{1}$$

The multiplicative form of (1) is appropriate for blackout event probabilities because it induces an accuracy requirement of similar multiplicative form in the risk of the blackout event, since risk is probability times cost. The confidence level is the probability that the accuracy will be satisfied. For example, a confidence level of 95% implies that if the estimation is performed 20 times, then one would expect by chance variations that on average 1 of the 20 estimations would give an inaccurate estimate $\hat{p}$ that does not satisfy (1).

TABLE I
MINIMUM NUMBER OF OCCURRENCES TO ESTIMATE EVENT PROBABILITY AS A FUNCTION OF CONFIDENCE LEVEL c AND ACCURACY FACTOR a

confidence	accuracy factor a					
level c	1.5	2	3	4	5	10
0.90	18	7	4	3	3	3
0.95	26	11	7	5	5	4
0.98	39	17	10	8	7	6

The minimum number of occurrences to achieve accuracy (1) at several confidence levels is given in Table I. For example, 11 occurrences are needed to estimate $\hat{p}$ within a factor of 2 with 95% confidence. It is easy to see how needing 11 occurrences affects the number n of blackout samples required. The average number of occurrences of an event with probability p in n samples is np. Therefore n and p must satisfy

$$np \ge 11.$$

That is, to accurately estimate a given p, at least $11/p$ samples are needed. Or, given n samples, then the smallest probability that can be accurately estimated is $11/n$.

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[1] "One swallow does not a summer make" Aristotle, 384-322 BC.

Another question is distinguishing whether an event would reliably have nonzero probability based on only a few occurrences of the event. If the event occurs, the best estimate of its probability is nonzero, but one can ask whether this conclusion would be reliably obtained from general samples. That is, one asks how many occurrences of an event are needed in order to conclude that the event has nonzero probability with a 95% confidence interval? The answer is that the event needs to occur 3 or more times.

Now consider reliably ranking the probability of 2 events based on n samples. Suppose that event 1 with probability p_1 happens h_1 times, and event 2 with probability p_2 happens h_2 times. The estimates for the probabilities of events 1 and 2 are

$$\hat{p}_1 = \frac{h_1}{n} \qquad \text{and} \qquad \hat{p}_2 = \frac{h_2}{n}.$$

If $h_2 \geq h_1$, when can we reliably conclude that $p_2 \geq p_1$? For a confidence level $c = 0.95$, Figure 1 shows pairs of integers (h_1, h_2) as grid intersection points. For a confidence level $c = 0.95$, we can conclude that $p_2 \geq p_1$ for all pairs of integers (h_1, h_2) above the solid line.

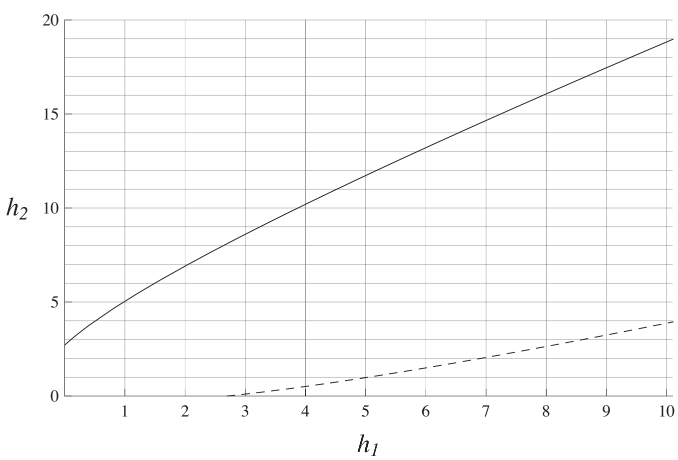

Fig. 1. Event 1 occurs h_1 times and event 2 independently occurs h_2 times. If (h_1, h_2) lies above the solid line, then probability of event 2 $\geq$ probability of event 1 with confidence level 0.95. If (h_1, h_2) lies below the dashed line, then probability of event 1 $\geq$ probability of event 2 with confidence level 0.95.

III. Deriving the Results

The key assumption is that the sample blackouts are independent. For example, simulated blackouts can often be assumed independent if each blackout is generated as a function of pseudorandom numbers generated in the computer. (Of course, having a sufficient number of samples of a rare event is not the only requirement for sound results. It is also necessary to include all the possibilities and to sample them uniformly so that there is no bias toward a particular subset of outcomes.)

The number of occurrences H of a rare event of probability p in n independent samples has distribution Binomial(n, p) with expected number of occurrences $EH = np = h$ and variance $\text{var}H = np(1-p)$. For rare events, $p << 1$ and $\text{var}H \approx np$. Moreover, for p small and n large, it is a very good approximation that Binomial$(n, p) \approx$ Normal$(np, np) =$ Normal(h, h), so that H is approximately distributed as Normal(h, h).

Our accuracy and reliability requirements are that the estimated probability $\hat{p}$ lies within a factor a of p with probability at least c; that is, $P\left[a^{-1}p \leq \hat{p} \leq ap\right] \geq c$. Since $\hat{p} = H/n$, equivalent requirements are $P\left[a^{-1}h \leq H \leq ah\right] \geq c$, and

$$P\left[(a^{-1}-1)\sqrt{h} \leq Z \leq (a-1)\sqrt{h}\right] \geq c, \qquad (2)$$

where $Z = (H-h)/\sqrt{h} \sim$ Normal$(0, 1)$. Given a and c, it is straightforward to numerically solve (2) with $\geq c$ replaced by $= c$ for a value of h and then round up to the nearest integer. Hence the results in Table I. A similar analysis is given in [4] for the different accuracy criterion that $(1-b)p \leq \hat{p} \leq (1+b)p$ for some constant b, and this is applied in [2].

How many occurrences h are needed to have less than $1-c$ chance of having had zero occurrences? Since $P[H=0] = (1-p)^n = 1-c$, $\quad -p \approx \ln(1-p) = (\ln(1-c))/n$ and $h \approx -\ln(1-c)$. For $c = 0.90$ or $c = 0.95$, $h \geq 3$ is needed. For $c = 0.98$, $h \geq 4$ is needed.

Let H_1 be the number of occurrences of event 1 and H_2 be the number of occurrences of event 2. According to the approximations above, $H_1 \sim$ Normal(h_1, h_1) and $H_2 \sim$ Normal(h_2, h_2). The ranking $p_2 \geq p_1$ is equivalent to $h_2 \geq h_1$, and the ranking $p_2 \geq p_1$ is correct with probability $P[H_2 - H_1 \geq 0]$.

First assume that H_1 is independent from H_2. Then $H_2 - H_1 \sim$ Normal$(h_2 - h_1, h_1 + h_2)$ and

$$P[H_2 - H_1 \geq 0] = P\left[Z \geq \frac{-(h_2-h_1)}{\sqrt{h_1+h_2}}\right], \qquad (3)$$

$$\text{where} \qquad Z = \frac{H_2 - H_1 - (h_2 - h_1)}{\sqrt{h_1+h_2}} \sim \text{Normal}(0,1).$$

The requirement is $P[H_2 - H_1 \geq 0] \geq c$. For a given confidence level c, we use (3) to solve $P[H_2 - H_1 \geq 0] = c$ to give h_2 as a function of h_1 as exemplified in Fig. 1 for $c = 0.95$.

Now assume that H_1 and H_2 are jointly normal with correlation coefficient ρ. Then

$$H_2 - H_1 \sim \text{Normal}(h_2 - h_1, h_1 + h_2 - 2\rho\sqrt{h_1 h_2}).$$

Following the previous computations yields (3) with the denominator $\sqrt{h_1+h_2}$ replaced by $\sqrt{h_1+h_2-2\rho\sqrt{h_1h_2}}$. It can be seen that a positive correlation $\rho > 0$ between H_1 and H_2 increases $P[H_2 - H_1 \geq 0]$ and so reduces the number of occurrences necessary for a given confidence level. Even if the exact correlation is not known, the sensitivity to correlation could readily be studied.

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