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Computing a Closest Bifurcation Instability in Multidimensional Parameter Space

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Summary. Engineering and physical systems are often modeled as nonlinear differential equations with a vector λ of parameters and operated at a stable equilibrium. However, as the parameters λ vary from some nominal value λ_0, the stability of the equilibrium can be lost in a saddle-node or Hopf bifurcation. The spatial relation in parameter space of λ_0 to the critical set of parameters at which the stable equilibrium bifurcates determines the robustness of the system stability to parameter variations and is important in applications. We propose computing a parameter vector λ_* at which the stable equilibrium bifurcates which is locally closest in parameter space to the nominal parameters λ_0. Iterative and direct methods for computing these locally closest bifurcations are described. The methods are extensions of standard, one-parameter methods of computing bifurcations and are based on formulas for the normal vector to hypersurfaces of the bifurcation set. Conditions on the hypersurface curvature are given to ensure the local convergence of the iterative method and the regularity of solutions of the direct method. Formulas are derived for the curvature of the saddle node bifurcation set. The methods are extended to transcritical and pitchfork bifurcations and parametrized maps, and the sensitivity to λ_0 of the distance to a closest bifurcation is derived. The application of the methods is illustrated by computing the proximity to the closest voltage collapse instability of a simple electric power system.

Key words. bifurcation, saddle node, Hopf, stability, robustness, optimization, numerical methods, transcritical, pitchfork, extended systems

1. Introduction

Consider a system modeled by smooth parametrized differential equations:

$$\dot{z} = f(z, \lambda), \qquad z \in \mathbb{R}^n, \qquad \lambda \in \mathbb{R}^m, \qquad m \geq 2. \tag{1.1}$$

We write x for a particular equilibrium of (1.1) and assume that x is asymptotically stable at the fixed parameter vector $\lambda_0 \in \mathbb{R}^m$. λ_0 arises in applications as the parameters at which the system is currently operated or nominal parameters for engineering design purposes. As λ varies in the parameter space $\mathbb{R}^m$, the equilibrium x varies in the state space $\mathbb{R}^n$ and may disappear or become unstable in a bifurcation. The set $\Sigma \subset \mathbb{R}^m$ of parameters at which x disappears or becomes unstable in a bifurcation determines the limits in parameter space at which the system may be stably operated at x and is of importance in applications. Σ is part of the bifurcation set and typically consists of hypersurfaces in $\mathbb{R}^m$ and their intersections.

Given the fixed parameter value λ_0, an obvious question concerns the spatial relation of λ_0 to Σ and in particular the points of Σ which are closest to λ_0. Indeed, the distance from λ_0 to the closest points of Σ measures the robustness of the system to parameter variations when it is operated at λ_0 and the directions of the closest points of Σ from λ_0 are "worst case" directions for parameter variations leading to disappearance or instability of x. There is an inherent difficulty in describing the geometry of multidimensional hypersurfaces such as Σ, and computing the points of Σ locally closest to a fixed parameter value λ_0 is a useful way to summarize the spatial relation of λ_0 to Σ while avoiding the difficult task of describing all of Σ. We call the bifurcations at the points of Σ locally closest to λ_0 "locally closest bifurcations."

Our computation of locally closest bifurcations is directed toward applications in which it is desirable to monitor and avoid bifurcation instability in a multidimensional parameter space. In applications such as the electric power system example of Sect. 8, the parameters λ vary slowly in time and the objective is to monitor the proximity to Σ so that corrective action can be taken if λ moves too close to Σ. In an engineering design application as in Mazzoleni and Dobson (1993), system (1.1) is being designed to be robust to parameter variations by ensuring that the nominal design parameter λ_0 is sufficiently far from Σ. In both types of applications, it is desirable to obtain both the direction in parameter space of a closest bifurcation and the distance in parameter space to this bifurcation. Indeed, if the distance to a closest bifurcation is dangerously small, then parameters should be changed (controlled or redesigned) to increase this distance. The optimum direction for this parameter change can be obtained from the sensitivity to λ_0 of the distance to a closest bifurcation as discussed in Sect. 9.

The key to the computation of a locally closest bifurcation is the normal vector to hypersurfaces of Σ and formulas for the normal vector are presented in Sect. 2. Section 3 expresses the conditions for a closest bifurcation in terms of the normal vector and curvature of Σ. In Sects. 4 and 5, we propose iterative and direct methods of computing locally closest bifurcations. The standard methods for computing points of Σ assume that λ varies along a given straight line in $\mathbb{R}^m$ so that the computation is reduced to the case of a one-dimensional parameter space. The formulas for the normal vector to Σ allow us to extend the standard methods to compute the points of Σ locally closest to λ_0. Conditions on the curvature of Σ are given to ensure the convergence of the iterative method and the regularity of solutions of the direct method.

We write x_* and λ_* for the equilibrium x and parameter λ at a bifurcation and $f_x|_* = f_x|_{(x_*,\lambda_*)}$ for the Jacobian f_x evaluated at the bifurcation. The hypersurfaces of Σ correspond to a real eigenvalue of $f_x|_*$ being zero (saddle node bifurcation) or a complex conjugate pair of eigenvalues on the imaginary axis (Hopf bifurcation). We write Σ^{sn} for the set of λ for which the Jacobian $f_x|_{(x,\lambda)}$ has a zero eigenvalue and Σ^{hopf} for the set of λ for which $f_x|_{(x,\lambda)}$ has a pair of imaginary eigenvalues. If system (1.1) is assumed to have no special restrictions or symmetries, then the saddle node and Hopf bifurcations are the only bifurcations generic in curves of parameters in $\mathbb{R}^m$ (Sotomayor 1973) and we can restrict our attention to these generically occurring bifurcations. Transcritical and pitchfork bifurcations arise in systems with special restrictions or symmetries and we consider computing closest transcritical and pitchfork bifurcations in Sect. 6. Section 6 also states the minor modifications needed to compute the closest bifurcations of parametrized maps. Section 7 derives formulas to compute the curvature of saddle node bifurcation hypersurfaces in Σ^{sn}. This paper presents and analyzes new methods for computing locally closest bifurcations but does not consider the detailed implementation or performance in particular applications. However, we illustrate the computation of a closest saddle node bifurcation in a simple electric power system in Sect. 8.

Most applications of bifurcation theory deal with a multidimensional parameter space by assuming that the parameters vary along a given straight line in order to reduce the problem to a one-parameter problem. This paper proposes one way to address bifurcation problems in a multidimensional parameter space. Other authors have addressed multidimensional parameter spaces in different ways and we briefly indicate their approaches.

Although the focus in this paper is codimension one bifurcations in a multidimensional parameter space, we note that it is natural to study bifurcations of higher codimension in a parameter space of dimension equal to the codimension of the bifurcation. For example, Spence and Werner (1982), Roose and Piessens (1985), and Griewank and Reddien (1990) compute cusps of codimension two in a two-dimensional parameter space using extended systems. Spence et al. (1990) and De Dier et al. (1990) study the B-points of codimension two in which a saddle node and Hopf bifurcation coincide in two-dimensional parameter space. Kaas-Petersen (1990) computes a codimension three event in which a curve of Hopf points has a loop that degenerates to a cusp in a three-dimensional parameter space.

One way to study the hypersurfaces of Σ is to compute curves in Σ. Rheinboldt (1982) studied three methods for computing curves of saddle node bifurcations in Σ^{sn} in a general parameter space with more than one parameter. Curves in Σ^{sn} of saddle node bifurcations in a two-dimensional parameter space were computed by Jepson and Spence (1985) and applied to perturbed bifurcations and isolas. De Dier et al. (1990) compute curves in Σ^{sn} and Σ^{hopf} and also curves of B-points in models of chemical reactions. Dai and Rheinboldt (1990) locally characterize Σ^{sn} with a minimal extended system and compute a simplicial approximation to Σ^{sn}.

Some previous work on the closest saddle node bifurcation has been done in the context of the stability of large scale electric power systems. (Most of the work addresses the saddle node bifurcation in the context of the existence of a solution of static equations at which it is feasible to operate the power system.) The idea of computing a closest instability in a real power injection parameter space is due to Galiana and Jarjis (1978). Using a hypothesis that Σ is convex, Galiana and Jarjis parameterize Σ with the normal vector N to Σ and define a measure D which is

the perpendicular distance from the operating real power injections λ_0 to the tangent hyperplane of Σ with normal N. Minimizing D with conjugate gradient methods yields a closest instability and this computation is illustrated in a 6-node power system. Jarjis and Galiana (1981) minimize a non-Euclidean distance to instability in a load power and voltage magnitude parameter space using Fletcher-Powell methods. Constrained minimization in the load power parameter space is also considered. More recently, the saddle node bifurcation has been associated with the increasingly serious problem of voltage collapse and blackout of the power system (Dobson and Chiang 1989; Fink 1989; Fink 1991). Jung et al. (1990) suggest a gradient projection optimization method to compute a closest saddle node bifurcation and Sekine et al. (1989) attempt to compute a closest saddle node bifurcation by gradient descent on the determinant of the Jacobian $f_x|_{(x,\lambda)}$. Dobson and Lu (1992a) use the direct and iterative methods of this paper to compute a closest saddle node bifurcation in a six-dimensional load power parameter space of a 5-node power system. The theory for the iterative method in this paper is summarized in Dobson (1992b).

2. Normal Vectors to Bifurcation Hypersurfaces in Parameter Space

This section derives formulas for the normal vectors to saddle node and Hopf bifurcation hypersurfaces in parameter space. The formulas are not new and are essentially well-known transversality conditions, but they are usually obtained for the case $m = 1$ of a one-dimensional parameter space and their useful interpretation as normal vectors to bifurcation hypersurfaces for $m \geq 2$ has not been emphasized.

Write Σ_T^{sn} for the set of $\lambda_* \in \Sigma^{sn}$ for which equation (1.1) has a saddle node bifurcation at (x_*, λ_*) with $f_x|_*$ having a unique simple zero eigenvalue and satisfying the transversality conditions

$$w_* f_\lambda|_* \neq 0, \tag{2.1}$$

$$w_* f_{xx}(v_*, v_*)|_* \neq 0, \tag{2.2}$$

where v_* and w_* are the right and left eigenvectors of $f_x|_*$ corresponding to the zero eigenvalue. $\lambda_* \in \Sigma_T^{sn}$ implies (Sotomayor 1973, Sect. 3.1; Chow and Hale 1982, Sect. 6.2) that there is an open set $V \ni \lambda_*$ such that $\Sigma_T^{sn} \cap V = \Sigma^{sn} \cap V$ and $\Sigma_T^{sn} \cap V$ is a smooth hypersurface and that the bifurcating equilibria near x_* are given by $u(\lambda_*')$ where u is a smooth function $u : \Sigma_T^{sn} \cap V \to \mathbb{R}^n$ and $u(\lambda_*) = x_*$. It follows that Σ^{sn} has a normal vector $N(\lambda_*)$ at $\lambda_* \in \Sigma_T^{sn}$ and the Gauss map $N : \Sigma_T^{sn} \to S^{m-1}$ is smooth. (S^{m-1} is the $m-1$ sphere of unit length vectors in $\mathbb{R}^m$.) Moreover,

$$N(\lambda_*) = \alpha w_* f_\lambda|_*, \tag{2.3}$$

where $\lambda_* \in \Sigma_T^{sn}$ and α is a nonzero real scale factor (also see formula (7.1)). $|\alpha|$ is chosen so that $|N(\lambda_*)| = 1$ and the sign of α is chosen so that changing λ in the direction of $N(\lambda_*)$ leads to the disappearance of x.

The normal vector of formula (2.3) is the natural geometric interpretation of the transversality condition (2.1) and follows from the development in Chow and Hale (1982, Sect. 6.2) or may be verified as follows (Dobson 1992a): For any $\lambda_* \in \Sigma_T^{sn}$ we have $f(x_*, \lambda_*) = 0$ where $x_* = u(\lambda_*)$. Let $d\lambda_*$ be any one-form in the cotangent space $T^*(\Sigma_T^{sn})_{\lambda_*}$. Then $0 = df = f_x|_* dx_* + f_\lambda|_* d\lambda_*$ and $w_* f_\lambda|_* d\lambda_* = -w_* f_x|_* dx_* = 0$. Since $d\lambda_*$ is arbitrary, $w_* f_\lambda|_*$ is a normal vector to Σ_T^{sn} at λ_*. Therefore $N(\lambda_*) = \alpha w_* f_\lambda|_*$ for some real α.

Write Σ_T^{hopf} for the set of $\lambda_* \in \Sigma^{hopf}$ for which equation (1.1) has a Hopf bifurcation at (x_*, λ_*) with $f_x|_*$ having a simple pair of eigenvalues $\pm j\omega_*$, $\omega_* \neq 0$ and all other eigenvalues with nonzero real parts and satisfying the transversality conditions (2.4,2.5) presented below. Since $f_x|_*$ is invertible, the implicit function theorem implies that there is a smooth function u defined in a neighborhood of λ_* with $u(\lambda_*) = x_*$, $u(\lambda) = x$ and $f(u(\lambda), \lambda) = 0$. Moreover, $u_\lambda = -f_x^{-1} f_\lambda$. There is also a smooth function μ defined in a neighborhood of λ_* with $\mu(\lambda_*) = j\omega_*$ and $\mu(\lambda)$ an eigenvalue of $f_x|_{(u(\lambda),\lambda)}$. Write v and w for the right and left complex eigenvectors of $f_x|_{(u(\lambda),\lambda)}$ corresponding to $\mu(\lambda)$; these eigenvectors are normalized according to $|v| = 1$ and $wv = 1$. Then v and w are smooth functions of λ in a neighborhood of λ_*. Write $v_* = v(\lambda_*)$ and $w_* = w(\lambda_*)$. The transversality conditions satisfied by $\lambda_* \in \Sigma_T^{hopf}$ are

$$c \neq 0, \tag{2.4}$$

$$D_\lambda(\mathrm{Re}\{\mu(\lambda)\}) \neq 0, \tag{2.5}$$

where c is a coefficient of cubic terms in the flow reduced to the center manifold and is a complicated function of triple derivatives of f (Guckenheimer and Holmes 1983; Golubitsky and Schaeffer 1985). $\lambda_* \in \Sigma_T^{hopf}$ implies (Sotomayor 1973, Sect. 3.4; Chow and Hale 1982, Sect. 9.5) that there is an open set $V \ni \lambda_*$ such that $\Sigma_T^{hopf} \cap V = \Sigma^{hopf} \cap V$ is a smooth hypersurface given by $\mathrm{Re}\{\mu(\lambda)\} = 0$. It follows that Σ^{hopf} has a normal vector $N(\lambda_*)$ at $\lambda_* \in \Sigma_T^{hopf}$ and the Gauss map $N : \Sigma_T^{hopf} \to S^{m-1}$ is smooth. Moreover,

$$N(\lambda_*) = \beta\, \mathrm{Re}\{w(-f_{xx} f_x^{-1} f_\lambda + f_{x\lambda})v\}|_*, \tag{2.6}$$

where β is a nonzero real scale factor. $|\beta|$ is chosen so that $|N(\lambda_*)| = 1$ and the sign of β is chosen so that changing λ in the direction of $N(\lambda_*)$ leads to the instability of the operating equilibrium.

To prove formula (2.6), we first compute the eigenvalue sensitivity μ_λ by following Horn and Johnson (1985). Differentiate

$$\mu = w f_x v$$

with respect to λ to obtain

$$\mu_\lambda = wD_\lambda(f_x)v + w_\lambda f_x v + w f_x v_\lambda = wD_\lambda(f_x)v + \mu D_\lambda(wv) = wD_\lambda(f_x)v.$$

Since $D_\lambda(f_x) = f_{xx} u_\lambda + f_{x\lambda} = -f_{xx} f_x^{-1} f_\lambda + f_{x\lambda}$,

$$\mu_\lambda = w(-f_{xx} f_x^{-1} f_\lambda + f_{x\lambda})v. \tag{2.7}$$

A normal vector to Σ_T^{hopf} at λ_* is $D_\lambda(\text{Re}\{\mu(\lambda)\})|_{\lambda_*} = \text{Re}\{\mu_\lambda\}|_{\lambda_*}$ so that (2.7) gives the required formula (2.6). A different scaling of formula (2.6) and similar computations appear in Golubitsky and Schaeffer (1985, pp. 352–355).

3. Closest Bifurcations and the Normal Vector and Curvature of Σ

We express conditions for a closest bifurcation at $\lambda_* \in \Sigma$ in terms of the normal vector and curvature of Σ at λ_*.

We write $\Sigma_T = \Sigma_T^{\text{sn}} \cup \Sigma_T^{\text{hopf}}$ and $N : \Sigma_T \to S^{m-1}$ for the Gauss map on Σ_T. If we define the distance to bifurcation $\Delta : \Sigma_T \to \mathbb{R}$ by $\Delta(\lambda'_*) = |\lambda'_* - \lambda_0|$, then a closest bifurcation to λ_0 is a local minimum of Δ. If Δ has a local minimum at λ_*, then λ_* is a critical point of Δ and $\lambda_* - \lambda_0$ is parallel to $N(\lambda_*)$. This observation explains the key role of the normal vector $N(\lambda_*)$ in computing a closest bifurcation.

The condition ensuring that a critical point λ_* of Δ is a strict local minimum is

$$|\lambda_* - \lambda_0| < (k^{\max})^{-1} \quad \text{if} \quad k^{\max} > 0, \tag{3.1}$$

where $k^{\max}$ is the maximum principal curvature of Σ_T at λ_* (Thorpe 1979, Chap. 16). That is, the radius $|\lambda_* - \lambda_0|$ of the sphere centered on λ_0 must be exceeded by the minimum radius of curvature $(k^{\max})^{-1}$ of Σ_T at λ_*. Note that the curvature condition (3.1) is always satisfied if λ_0 is close enough to Σ_T or if $k^{\max}$ is negative.

The curvature of Σ_T at λ_* can be stated in terms of N (Thorpe 1979; Dai and Rheinboldt 1990). The second fundamental form II: $T\Sigma_T \times T\Sigma_T \to \mathbb{R}$ is the quadratic form given by

$$\text{II}(a, a) = a^T N_{\lambda_*} a. \tag{3.2}$$

The eigenvalues of $\text{II}|_*$ are the principal curvatures of Σ_T at λ_* and we write $k^{\max}$ for the maximum principal curvature and $k^{\min}$ for the minimum principal curvature. In the case of the saddle node bifurcation, Sect. 7. presents formulas for the curvature II of Σ^{sn} so that $k_{\max}$ of (3.1) may be computed.

4. Iterative Method

This section describes an iterative method for computing a locally closest saddle node or Hopf bifurcation. There are well known direct, continuation, and iterative methods for computing the closest bifurcation to λ_0 along a one-dimensional ray in parameter space based at λ_0, e.g. Seydel (1988) or Mittelman and Weber (1980). Any of these methods is used to compute the nearest bifurcation at, say, λ_*, along some ray based at λ_0. The normal vector $N(\lambda_*)$ is then computed using the formulas of Sect. 2 and a new ray based at λ_0 in the same direction as $N(\lambda_*)$ is chosen. This procedure is iterated until the successive rays tend to a fixed ray and the corresponding λ_* tends to the parameter value of a locally closest bifurcation. We analyze the iteration near its fixed points and show that exponentially stable fixed points λ_* are the parameters of locally closest bifurcations. Moreover, if $\lambda_* \in \Sigma_T$ is the parameter of a locally closest bifurcation and Σ_T is not "too concave" at λ_*, then the iteration has a corresponding

exponentially stable fixed point. For convenience, we consider closest saddle node bifurcations; the discussion applies to closest Hopf bifurcations with the substitution of Σ^{hopf} for Σ^{sn} and A^{hopf} for A^{sn}.

Each ray in parameter space starting at λ_0 is specified by a unit vector $n \in S^{m-1}$ so that the ray is $\text{Ray}(n) = \{\lambda_0 + tn \mid t \geq 0\}$. We define a function $\rho(n)$ which yields the parameter value λ_* of the closest bifurcation along $\text{Ray}(n)$. More precisely, if there is a smallest positive t_* with $\lambda_* = \lambda_0 + t_* n \in \Sigma^{\text{sn}}$, we define the function ρ at n by $\rho(n) = \lambda_*$. Thus ρ maps a subset of S^{m-1} into Σ^{sn}. Any of the standard methods for finding saddle node bifurcations along rays may be used to compute ρ (Seydel 1988; Mittelman and Weber 1980).

Define the set A^{sn} to be the $n \in S^{m-1}$ such that ρ is defined on n and $\rho(n) = \lambda_* \in \Sigma_T^{\text{sn}}$ and $\text{Ray}(n)$ is transverse to Σ_T^{sn} at λ_*. Define a map $h : A^{\text{sn}} \to S^{m-1}$ by $h(n) = N(\rho(n))$. The lemmas of Appendix 1 prove that A^{sn} is open and that h is smooth. We iterate h to try to find the direction from λ_0 of a locally closest saddle node bifurcation: Start with some initial direction $n_0 \in A^{\text{sn}}$ and obtain n_r, $r = 1, 2, 3, \ldots$ by

$$n_{r+1} = h(n_r) = N(\rho(n_r)). \tag{4.1}$$

Iteration (4.1) fails if $n_r \notin A^{\text{sn}}$ for some r.

At each step of the iteration the updated direction n_{r+1} is normal to both $\Sigma^{\text{sn}}|_{\rho(n_r)}$ and its tangent hyperplane $T\Sigma^{\text{sn}}|_{\rho(n_r)}$. It follows that the updated direction n_{r+1} points toward the point on $T\Sigma^{\text{sn}}|_{\rho(n_r)}$ which is globally closest to λ_0. Thus iteration (4.1) may be understood as globally minimizing the distance from λ_0 to a series of tangent hyperplane approximations to Σ.

The fixed points of h are exactly those points $n_* \in A^{\text{sn}}$ for which $\rho(n_*) = \lambda_*$ is a critical point of $\Delta(\lambda_*) = |\lambda_* - \lambda_0|$. It is straightforward to deduce the local behavior of h near its fixed points from its linearization h_n: Since $TS^{m-1}|_{n_*}$ is parallel to $T\Sigma^{\text{sn}}|_{\lambda_*}$, $\rho_n|_{n_*} : TS^{m-1}|_{n_*} \to T\Sigma^{\text{sn}}|_{\lambda_*}$ is the uniform dilation with factor $|\lambda_* - \lambda_0|$ and

$$h_n|_{n_*} = N_{\lambda_*} \rho_n|_{n_*} = \text{II}|_* |\lambda_* - \lambda_0|.$$

Hence the fixed point n_* is exponentially stable iff

$$k^{\max}|\lambda_* - \lambda_0| < 1 \tag{4.2}$$

and

$$k^{\min}|\lambda_* - \lambda_0| > -1, \tag{4.3}$$

where $k^{\max}$ and $k^{\min}$ are the maximum and minimum eigenvalues of $\text{II}|_*$ and the maximum and minimum principal curvatures of Σ^{sn} at λ_*. If λ_* is a strict local minimum of Δ then condition (4.2) holds (see the identical condition (3.1) and discussion in Sect. 3) and h converges to the fixed point n_* iff condition (4.3) is satisfied; that is, if Σ^{sn} is not "too concave" at λ_*. The convergence is geometric with factor $\max\{|k^{\max}|, |k^{\min}|\}|\lambda_* - \lambda_0|$ and is asymptotically along a direction of maximum absolute curvature. The iteration converges quickly if the curvature of Σ^{sn} is small or λ_0 is close to Σ^{sn}. If λ_* is not a local minimum of Δ and $k^{\max}|\lambda_* - \lambda_0| > 1$ then the iteration diverges from the fixed point n_*.

We emphasize that the iterative method is easy to implement; all that is required is iteration of any method to compute the nearest bifurcation along a given ray and the evaluation of a formula for the normal vector $N(\lambda_*)$.

5. Direct Method

This section shows how to construct direct methods for computing locally closest saddle node or Hopf bifurcations which are regular at solutions.

A saddle node bifurcation at (x_*, λ_*) with $\lambda_* \in \Sigma_T^{\text{sn}}$ satisfies $f(x_*, \lambda_*) = 0$ and $w_* f_x|_* = 0$. The left eigenvector w_* can be scaled to satisfy $w_* f_\lambda|_* (w_* f_\lambda|_*)^T = 1$. If λ_* is a locally closest bifurcation to λ_0, then $\lambda_* - \lambda_0$ is parallel to the normal vector $N(\lambda_*) = w_* f_\lambda|_*$ of Σ_T^{sn} at λ_* so that $-k_*(\lambda_* - \lambda_0)^T + w_* f_\lambda|_* = 0$ for some nonzero $k_* \in \mathbb{R}$. Consider the closest saddle node equations

$$F^{\text{sn}}(x, \lambda, w, k) = 0, \tag{5.1}$$

$$F^{\text{sn}} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{2n+m+1},$$

where

$$F^{\text{sn}}(x, \lambda, w, k) = \begin{cases} f(x, \lambda) & (5.1.1) \\ w f_x & (5.1.2) \\ w f_\lambda (w f_\lambda)^T - 1 & (5.1.3) \\ k(\lambda - \lambda_0)^T - w f_\lambda & (5.1.4) \end{cases}$$

A locally closest saddle node bifurcation satisfies $F^{\text{sn}}(x_*, w_*, \lambda_*, k_*) = 0$. However, any saddle node bifurcation with $\lambda_* \in \Sigma_T^{\text{sn}}$ and a critical point of $\Delta(\lambda_*') = |\lambda_*' - \lambda_0|$ at λ_* will satisfy $F^{\text{sn}}(x_*, w_*, \lambda_*, k_*) = 0$. The curvature condition (3.1) is required to ensure that λ_* is a local minimum of $\Delta(\lambda_*')$. For practical calculation it is often convenient to replace expression (5.1.3) by the simpler expression $wq - 1$ where q is a fixed vector (Moore and Spence 1980; Spence and Werner 1982; Seydel 1988).

Equations (5.1) are the extended system equations for saddle node bifurcations (Moore and Spence 1980; Spence and Werner 1982; Seydel 1988) (with left eigenvectors instead of right eigenvectors) generalized to a parameter $\lambda \in \mathbb{R}^m$ by the addition of the m equations (5.1.4). In the case of a scalar parameter ($m = 1$), (5.1.4) yields a trivial equation for k which may be omitted and the equations reduce to the extended system equations.

Equations (5.1) may also be related to an optimization similar to that proposed by Jung et al. (1990) in the context of voltage stability in power systems: Minimize $\frac{1}{2}|\lambda - \lambda'|^2$ subject to the zeroing of the expressions (5.1.1), (5.1.2), (5.1.3) for $\lambda \in \mathbb{R}^m$. (Jung et al. (1990) use a right eigenvector instead of the left eigenvector w.) The Lagrangian is

$$L = \tfrac{1}{2}|\lambda - \lambda'|^2 + m_1 f + m_2 (w f_x)^T + m_3 (w f_\lambda (w f_\lambda)^T - 1). \tag{5.2}$$

In the voltage stability problem the parameters λ are load powers and these enter linearly into (1.1) so that f_x and w are not explicit functions of λ. In this case the derivative of L with respect to λ is

$$L_\lambda = (\lambda - \lambda')^T + m_1 f_\lambda. \tag{5.3}$$

Now L_λ reduces to (5.1.4) with Lagrange multiplier $m_1 = -w/k$.

If Newton-type methods are used to solve (5.1) then the asymptotic convergence to a solution is quadratic, which is an improvement over the geometric convergence of the iterative method. However, Newton-type methods require good initial estimates of the solution. Another disadvantage of the direct method is the burden of checking that the solution of (5.1) yields a locally closest bifurcation with the curvature condition (3.1).

Extended system equations to compute Hopf bifurcations may be similarly generalized to compute locally closest Hopf bifurcations:

$$F^{\text{hopf}}(x, \lambda, \omega, v_1, v_2, w_1, w_2, u, k) = 0, \tag{5.4}$$

$$F^{\text{hopf}} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^{5n} \times \mathbb{R} \to \mathbb{R}^{6n+m+2},$$

where

$F^{\text{hopf}}(x, \lambda, \omega, v_1, v_2, w_1, w_2, u, k) =$

$$\begin{cases} f(x, \lambda) & (5.4.1) \\ f_x|_* v_1 - \omega v_2 & (5.4.2) \\ f_x|_* v_2 + \omega v_1 & (5.4.3) \\ q^T v_1 & (5.4.4) \\ v_1^T v_1 - 1 & (5.4.5) \\ (w_1 f_x|_* - \omega w_2)\pi & (5.4.6) \\ (w_2 f_x|_* + \omega w_1)\pi & (5.4.7) \\ w_1 v_1 + w_2 v_2 - 1 & (5.4.8) \\ w_1 v_2 - w_2 v_1 & (5.4.9) \\ f_x|_* u - f_\lambda|_* & (5.4.10) \\ k(\lambda - \lambda_0)^T - w_1 (f_{xx} u + f_{x\lambda})|_* v_1 - w_2 (f_{xx} u + f_{x\lambda})|_* v_2 & (5.4.11) \end{cases}$$

where q is a fixed vector and π is a projection of $\mathbb{R}'^n$ onto $\mathbb{R}'^{(n-1)}$ (prime denotes dual space). π is assumed to satisfy the generic condition that its kernel satisfies $\ker\pi \cap \text{range}(f_x^T|_* - i\omega) = 0$ when π is considered as a map $\mathbb{C}'^n$ onto $\mathbb{C}'^{(n-1)}$. Equations (5.4.1)–(5.4.5) are the extended system equations of Roose and Hlavecek (1985). The normalization (5.4.4), (5.4.5) of the complex eigenvector $v_1 + iv_2$ is chosen so that saddle node bifurcations do not satisfy these equations (Roose and Hlavecek 1985). Equations (5.4.6)–(5.4.10) compute the additional vectors w_1, w_2, u needed

in (5.4.11) and (5.4.11) is the condition that $\lambda_* - \lambda_0$ is parallel to the normal vector $N(\lambda_*)$. Equations (5.4) have dimension $6n + m + 2$ so that this direct method for computing Hopf bifurcations may be too cumbersome to be useful if n is large.

Appendix 2 proves that the closest bifurcation systems above are regular at solutions if $\lambda_* \in \Sigma_T$ and the curvature condition (3.1) is satisfied. This regularity implies that the solutions are isolated and Newton type methods may be used to solve the equations with quadratic convergence.

6. Closest Transcritical and Pitchfork Bifurcations and Closest Bifurcations of Maps

If the system (1.1) always has an equilibrium $x = 0$, then saddle node bifurcations of the equilibrium $x = 0$ cannot occur and the transcritical bifurcation of x occurs generically in one-parameter families. (It may be necessary to change coordinates so that the trivial solution is at zero.) Also if the system has the symmetry $f(-x, \lambda) = -f(x, \lambda)$ for all $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$, then there is always an equilibrium at $x = 0$, saddle node and transcritical bifurcations do not occur, and the pitchfork bifurcation occurs generically in one-parameter families (Guckenheimer and Holmes 1983). This section describes formulas for normal vectors to bifurcation set hypersurfaces and methods for computing a closest bifurcation for the transcritical and pitchfork bifurcations. Closest bifurcations for parametrized maps are briefly considered.

We first consider the transcritical bifurcation. Suppose system (1.1) satisfies $f(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}^m$. We assume that the system is operated at $x = 0$ and that zero is asymptotically stable for $\lambda = \lambda_0$. Write Σ^{tc} for the set of λ for which the Jacobian $f_x|_{(0,\lambda)}$ has a zero eigenvalue and write Σ_T^{tc} for the set of $\lambda_* \in \Sigma^{tc}$ for which $f_x|_*$ has a unique simple zero eigenvalue with corresponding right and left eigenvectors v_*, w_* and satisfying the transversality conditions (2.2) and

$$w_* f_{x\lambda}|_* v_* \neq 0. \tag{6.1}$$

There is a smooth function μ defined in a neighborhood of λ_* with $\mu(\lambda_*) = 0$ and $\mu(\lambda)$ an eigenvalue of $f_x|_{(0,\lambda)}$. $\lambda_* \in \Sigma^{tc}$ implies (Sotomayor 1973, Sect. 3.3) that there is an open set $V \ni \lambda_*$ such that $\Sigma_T^{tc} \cap V = \Sigma^{tc} \cap V$ is a smooth hypersurface given by $\mu(\lambda) = 0$. It follows that Σ^{tc} has a normal vector $N(\lambda_*)$ at $\lambda_* \in \Sigma_T^{tc}$ and the Gauss map $N : \Sigma_T^{tc} \to S^{m-1}$ is smooth. Moreover,

$$N(\lambda_*) = \beta\, w_* f_{x\lambda}|_* v_*, \tag{6.2}$$

where β is a nonzero real scale factor and v_* and w_* are normalized so that $w_* v_* = 1$. $|\beta|$ is chosen so that $|N(\lambda_*)| = 1$ and the sign of β is chosen so that changing λ in the direction of $N(\lambda_*)$ leads to the instability of zero. Formula (6.2) is proved in the same way as the formula (2.6) for the Hopf bifurcation.

The formula for the normal vector for the pitchfork bifurcation hypersurface is also (6.2) and is derived similarly except that the transversality conditions for the pitchfork are (6.1) and a condition on triple derivatives of f (Guckenheimer and Holmes 1983).

Formula (6.2) together with any method for computing a transcritical or pitchfork bifurcation along a given ray are sufficient to implement the iterative method. A direct

method to compute transcritical or pitchfork bifurcations is

$$F^{tc}(v, \lambda, w, k) = 0, \tag{6.3}$$

$$F^{tc} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{2n+m+1},$$

where,

$$F^{tc}(v, w, \lambda, k) = \begin{cases} f_x|_{(0,\lambda)} v & (6.3.1) \\ v^T v - 1 & (6.3.2) \\ w f_x|_{(0,\lambda)} \pi & (6.3.3) \\ wv - 1 & (6.3.4) \\ k(\lambda - \lambda_0)^T - w f_{x\lambda}|_{(0,\lambda)} v & (6.3.5) \end{cases}$$

and π is a projection of $\mathbb{R}^{\prime n}$ onto $\mathbb{R}^{\prime(n-1)}$ whose kernel satisfies $\ker\pi \cap \text{range} f_x^T|_* = 0$. Appendix 2 proves that (6.3) is regular at solutions satisfying (6.1).

It is also straightforward to compute the closest bifurcations of smooth parametrized maps $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ because the bifurcation theory of maps parallels the bifurcation theory of differential equations (Guckenheimer and Holmes 1983; Sotomayor 1973). Saddle node bifurcation of g occurs when g_x has a simple eigenvalue one and the formulas and methods for saddle node bifurcations of f apply to saddle node bifurcations of g if $g(x, \lambda) - x$ is written for f throughout. For example, f_x becomes $g_x - I$ and $f_{x\lambda}$ becomes $g_{x\lambda}$. Hopf or period doubling bifurcations occur when g_x has a simple pair of complex eigenvalues on the unit circle or a simple eigenvalue -1 respectively (Guckenheimer and Holmes 1983; Sotomayor 1973). The formulas and methods for the Hopf bifurcation of f apply to Hopf or period doubling bifurcations of a map g if $g(x, \lambda) - x$ is written for f throughout except that the Hopf theory for maps requires the additional genericity conditions $\mu^2 \neq 1$ and $\mu^3 \neq 1$.

7. Formulas for the Saddle Node Bifurcation Set Curvature

This section derives computable formulas for the curvature II of Σ_T^{sn}. (Recall that the direct method requires II and k_{max} to be computed for use in (3.1).) Numerical methods for computing hypersurface curvature are described in Rabier and Rheinboldt (1990).

The first task is to deal more carefully with the specification of vectors in the range space of the Gauss map N. Write i_1 for the injection of the unit sphere S^{m-1} into $\mathbb{R}^m$ and i_2 for the injection of Σ_T^{sn} into $\mathbb{R}^m$ and i_3 for the injection of the unit sphere $S^{\prime m-1}$ into the dual space $\mathbb{R}^{\prime m}$. Note that we can identify $Di_1|_{N(\lambda_*)^T}$ with $Di_2|_{\lambda_*}$ since the tangent spaces $TS^{m-1}|_{N(\lambda_*)^T}$ and $T\Sigma_T^{sn}|_{\lambda_*}$ are parallel. ($N(\lambda_*)^T \in S^{m-1}$ is the vector dual to $N(\lambda_*) \in S^{\prime m-1} \subset \mathbb{R}^{\prime m}$.) Moreover, Di_3^{-1} is the projection $T\mathbb{R}^{\prime m} \to TS^{\prime m-1}$ along $N(\lambda_*)$ and its dual is the injection $Di_1 : TS^{m-1} \to T\mathbb{R}^m$. Formula (2.3) describes $N(\lambda_*)$ by a vector in $\mathbb{R}^{\prime m}$; the corresponding formula describing $N(\lambda_*)$ by a vector in $S^{\prime m-1}$ is

$$N(\lambda_*) = i_3^{-1}(\alpha w_* f_\lambda|_*). \tag{7.1}$$

In this section, it is convenient to assume that w_* is scaled and oriented so that $|(w_* f_\lambda|_*)| = 1$ and $\alpha = 1$. Then

$$N_{\lambda_*} = Di_3^{-1}\left((w_* f_\lambda)_{\lambda_*}\right) = (w_* f_\lambda)_{\lambda_*} Di_1 = (w_* f_\lambda)_{\lambda_*} Di_2$$
$$= w_{*\lambda_*} f_\lambda Di_2 + w_* f_{\lambda\lambda_*} Di_2.$$

Use $f_{\lambda_*} = f_\lambda Di_2$ to obtain

$$\text{II} = N_{\lambda_*} = w_{*\lambda_*} f_{\lambda_*} + w_* f_{\lambda_*\lambda_*}. \tag{7.2}$$

Now we seek to express the term involving $w_{*\lambda_*}$ in terms of more readily computable quantities. Differentiating $f(u(\lambda_*), \lambda_*) = 0$ with respect to λ_* yields

$$f_x u_{\lambda_*} + f_{\lambda_*} = 0 \tag{7.3}$$

so that (7.2) becomes

$$\text{II} = -w_{*\lambda_*} f_x u_{\lambda_*} + w_* f_{\lambda_*\lambda_*}. \tag{7.4}$$

Differentiating $w_* f_x = 0$ with respect to λ_* gives

$$w_{*\lambda_*} f_x + w_* f_{xx} u_{\lambda_*} + w_* f_{x\lambda_*} = 0 \tag{7.5}$$

so that (7.4) becomes

$$\text{II} = w_* f_{xx}(u_{\lambda_*}, u_{\lambda_*}) + w_* f_{x\lambda_*} u_{\lambda_*} + w_* f_{\lambda_*\lambda_*}. \tag{7.6}$$

It remains to calculate the quantity u_{λ_*} appearing in (7.6). u_{λ_*} satisfies (7.3) but this is not sufficient to obtain u_{λ_*} since $f_x|_*$ has rank $n - 1$. Differentiating $w_* f_x v_* = 0$ with respect to λ_* gives

$$w_* f_{xx}(u_{\lambda_*}, v_*) + w_* f_{x\lambda_*} v_* = 0 \tag{7.7}$$

which, together with (7.3), is sufficient to calculate u_{λ_*}. A numerical example of the computation of II for Σ^{sn} five-dimensional is given in Dobson and Lu (1992a).

8. Illustrative Computation in a Simple Electric Power System

In large scale electric power systems, the system state includes the circuit node voltages and the load powers are often modeled as slowly varying parameters. Saddle node bifurcation of the operating equilibrium of the power system is associated with a "voltage collapse" in which system voltages decline catastrophically (Dobson and Chiang 1989; Fink 1989; Fink 1991). Since voltage collapse causes blackouts, there is a strong economic incentive to avoid saddle node bifurcation by monitoring the proximity of the current load powers relative to Σ^{sn} and taking corrective action if the load powers are too close to Σ^{sn}. (Note that load powers are controlled by the consumers and are generally not controlled by the utility except when selectively disconnected in emergencies.) The distance in the load power parameter space from the

current operating load powers to a closest saddle node bifurcation is a measure of the proximity of the system to voltage collapse.

We sketch the computation of a closest saddle node bifurcation in a simple electric power system example. The example has a two-dimensional parameter space which is convenient for easy illustration. However, we expect to use higher dimensional parameter spaces in most applications. For example, Dobson and Lu (1992a) use the direct and iterative methods to compute a closest saddle node bifurcation in the six-dimensional load power parameter space of an electric power system.

Consider a simple power system model of a generator slack bus, lossless line and a load with real power P and reactive power Q (Bergen 1986). The system state $x = (\alpha, V)$ where $Ve^{i\alpha}$ is the load voltage phasor. The static (load flow) equations governing the system equilibria are

$$0 = -4V \sin\alpha - P,$$
$$0 = -4V^2 + 4V\cos\alpha - Q.$$

(The generator voltage is 1.0 and the line admittance is 4.0. All quantities are in per unit except that angles are in radians.) The parameter vector $\lambda = (P, Q)$ contains the real and reactive powers consumed by the load. Note that the parameters enter linearly and that f_λ is the negative of the 2×2 identity matrix. The system is operated at $\lambda_0 = (P_0, Q_0) = (0.5, 0.3)$ (see Fig. 1) and the corre-

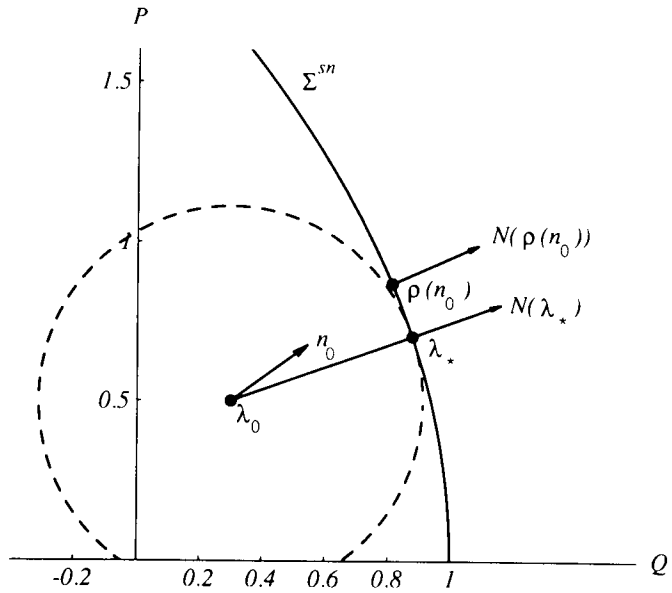

Fig. 1. Parameter space of a two-bus power system.

Table 1. Convergence of iterative method.

r	n_r	$\rho(n_r)$
0	(0.585, 0.811)	(0.869, 0.811)
1	(0.398, 0.917)	(0.744, 0.862)
2	(0.349, 0.937)	(0.713, 0.873)
3	(0.336, 0.942)	(0.705, 0.876)
4	(0.333, 0.943)	(0.703, 0.876)
5	(0.332, 0.943)	(0.703, 0.877)
6	(0.331, 0.943)	(0.703, 0.877)

sponding equilibrium is $x = (\alpha_0, V_0) = (-0.138, 0.908)$. The closest bifurcation at $\lambda_* = (P_*, Q_*) = (0.703, 0.877)$ was computed using the iterative method (4.1) (see Table 1). The same answer is obtained in Dobson and Lu (1992a) by solving the direct method equations (5.1) by Newton's method. The normal vector to Σ at λ_* is $N(\lambda_*) = (0.331, 0.943)$ and the minimum distance to bifurcation is $|\lambda_* - \lambda_0| = 0.611$.

As discussed in Sect. 4, the convergence of the iterative method to the solution demonstrates that λ_* is a locally closest bifurcation. This may also be checked by computing the curvature $k_{\max} = \mathrm{II}_* = 0.420$ of Σ^{sn} at λ_* using the formulas of Sect. 7. The curvature of the dashed circle in Fig. 1 is $k_* = 1.636$ and since k_* exceeds $k_{\max}$, condition (3.1) is satisfied, confirming that the solution is a locally closest bifurcation.

9. Sensitivity to λ_0 of the Distance to Bifurcation

If the distance $\Delta = |\lambda_* - \lambda_0|$ to a bifurcation instability is dangerously small, it is desirable to change λ_0 to increase Δ. The optimum direction of first order changes in λ_0 to increase Δ is obtained from the sensitivity Δ_{λ_0}. For example, Dobson and Lu (1992b) assume a fixed direction n for one-parameter variation along $\mathrm{Ray}(n) = \{\lambda_0 + tn \mid t \geq 0\}$ and compute the sensitivity

$$\Delta_{\lambda_0} = -(N(\lambda_*) \cdot n)^{-1} N(\lambda_*)$$

and use this to compute control actions to avoid saddle node bifurcation instability of an electric power system. (The parameter space can be augmented with system controls if they are not already considered as system parameters.)

It is also straightforward to compute the sensitivity to λ_0 of the distance Δ to a locally closest bifurcation. It is convenient to regard $\Delta(\lambda'_*, \lambda'_0) = |\lambda'_* - \lambda'_0|$ as a function $\Delta : \Sigma_T \times \mathbb{R}^m \to \mathbb{R}$. Suppose $\lambda_* \in \Sigma_T$ is a strict local minimum of $\Delta(\lambda'_*, \lambda_0)$ with respect to λ'_*. Then

$$\Delta_{\lambda'_*}|_{(\lambda_*, \lambda_0)} = 0 \tag{9.1}$$

and

$$\Delta_{\lambda'_*\lambda'_*}|_{(\lambda_*, \lambda_0)} = \Delta^{-1} I - \mathrm{II}_{\lambda_*}. \tag{9.2}$$

((9.2) may be obtained from $\Delta_{\lambda'_*} = \Delta^{-1}[(\lambda'_* - \lambda_0) - \{(\lambda'_* - \lambda_0).N(\lambda'_*)\}N(\lambda'_*)]$.) Equation (9.2) and the curvature condition (3.1) imply that $\Delta_{\lambda'_*\lambda'_*}|_{(\lambda_*, \lambda_0)}$ is positive definite and invertible.

Since $\Delta_{\lambda'_*\lambda'_*}|_{\lambda_*}$ is invertible, the implicit function theorem and (9.1) imply that there is a neighborhood $U \ni \lambda_0$ and a smooth function $\Lambda_* : U \to \Sigma_T$ such that $\Delta_{\lambda'_*}|_{(\Lambda_*(\lambda'_0), \lambda'_0)} = 0$ for $\lambda'_0 \in U$. Since $\mathrm{II}_{\lambda'_*}$ is continuous, U may be shrunk so that $\Lambda_*(\lambda'_0)$ is also a strict local minimum of $\Delta(\lambda'_*, \lambda'_0)$ with respect to λ'_*. Then $\Delta^0(\lambda'_0) = \Delta(\Lambda_*(\lambda'_0), \lambda'_0)$ gives the distance to a locally closest bifurcation as a function of $\lambda'_0 \in U$ and the required sensitivity is given by

$$\Delta^0_{\lambda'_0}|_{\lambda_0} = \Delta_{\lambda'_0}|_{(\lambda_*, \lambda_0)} + \Delta_{\lambda'_*}|_{(\lambda_*, \lambda_0)} \Lambda_{*\lambda'_0}|_{\lambda_0} = -N(\lambda_*).$$

This conclusion is also stated in Galiana and Jarjis (1978). The geometric content is clear: the optimum direction to move away from a closest bifurcation is antiparallel to the normal vector $N(\lambda_*)$.

10. Conclusions

We address in this paper the general problem of finding bifurcations of a stable equilibrium which are closest to a given parameter vector λ_0 in a high dimensional parameter space. Reliable and efficient solution of this problem would be especially useful in engineering applications in which the distance to a bifurcation instability should be monitored during operation or design of the system.

We show how to construct novel iterative and direct methods to compute bifurcations which are locally closest to λ_0. Methods for closest saddle node, Hopf, transcritical, and pitchfork bifurcations are proposed, and methods for closest bifurcations of maps are sketched. These bifurcations are codimension one and are assumed to be generic. The set of parameter values Σ at which the system loses stability has the form of hypersurfaces in parameter space. We give conditions on the curvature of Σ for convergence of the iterative method and regularity of solutions of the direct method. Particularly useful is the property of the iterative method that if it converges exponentially to a solution λ_*, then λ_* specifies a locally closest bifurcation and second order curvature conditions need not be checked. However, the iterative method does not converge to a locally closest bifurcation specified by λ_* if Σ is sufficiently concave at λ_*. The direct method requires good initial solution estimates and the curvature of Σ to be checked at the solution. We supply suitable formulas for the curvature of Σ in the case of a closest saddle node bifurcation. These formulas are also of independent interest since the curvature and the normal vector of Σ describe the local geometry of Σ to second order. Both the iterative and direct methods are extensions of standard one-parameter methods to the multiparameter case. In particular, the iterative method is easy to implement since it consists of repeatedly computing one-parameter bifurcation problems and normal vectors to Σ.

The paper does not address the effectiveness of the proposed methods in specific applications but does provide some basic theory to show that the proposed methods are sensible. The proposed methods have been used to find the closest saddle node bifurcation instabilities of electric power systems with small numbers of buses and the

initial results are good (Dobson and Lu 1992a). If λ_0 is close to several portions of Σ, it would be desirable to compute a locally closest saddle node bifurcation on each of the portions of Σ. This could be attempted by searching in several initial directions with the proposed methods. More needs to be known about the structure of Σ specific to each application in order that these methods can be shown effective or improved to take advantage of this structure.

Computing a closest bifurcation might also be useful in finding realistic sets of parameters at which mathematical models bifurcate. (This can be difficult by trial and error or by intuition if there are many parameters.) If realistic parameters λ_0 are chosen, then a closest bifurcation allowing many parameters to vary yields a bifurcation which is least unrealistic.

Little has been done on computing the proximity to codimension one bifurcation instabilities in higher dimensional parameter spaces while allowing all the parameters to vary independently, despite the practical importance of this computation. The iterative and direct methods we propose appear to be new and our aim here is to derive and explain the methods and provide some theoretical foundation so that their testing and development in applications is encouraged.

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Appendix 1: Smoothness of the Iterative Map h

We prove that the iterative map h is smooth and that its domain A^{sn} is open for the saddle node and Hopf bifurcations. The proof for the transcritical and pitchfork bifurcations is exactly parallel to the proof for the Hopf bifurcation.

Lemma 1. *Suppose $n \in A^{sn}$ and write $\rho(n) = \lambda_* = \lambda_0 + nt_* \in \Sigma_T^{sn}$ and V for the neighborhood of λ_* such that $\Sigma^{sn} \cap V$ is a smooth hypersurface containing λ_*. Then there is a neighborhood U of n with $U \subset A^{sn}$, $\rho(U) \subset \Sigma^{sn} \cap V$ and ρ smooth on U. It follows that A^{sn} is open and that h is smooth on U.*

Proof. Since the saddle node bifurcation at λ_* satisfies the transversality conditions (2.1),(2.2), there is a ball $B \subset V$ centered on λ_* of radius 2ϵ such that there is a saddle node bifurcation of x for $\lambda \in B$ iff $\lambda \in B \cap \Sigma^{sn}$. Moreover, $B \cap \Sigma^{sn}$ divides B into two connected components B^+, B^- with x simple for $\lambda \in B^-$, and no equilibrium near x_* for $\lambda \in B^+$ (Sotomayor 1973, Sect. 3.1; Chow and Hale 1982, Sect. 6.2). For each $t \in [0, t_* - \epsilon]$, the definition of ρ ensures that x is simple if $\lambda = \lambda_0 + nt$ and $t \in [0, t_* - \epsilon]$. Indeed x is simple for λ in a ball centered on $\lambda_0 + nt$. Since $[0, t_* - \epsilon]$ is compact, there is an $\epsilon_1 > 0$ such that x is simple for $\lambda \in V_1 = \{ \lambda \mid |\lambda - (\lambda_0 + tn)| < \epsilon_1 \text{ for some } t \in [0, t_* - \epsilon] \}$. Now choose an open set $U \subset S^{m-1}$, $U \ni n$ such that for $n' \in U$, $\lambda_0 + n't \in V_1 \cup B$ for $t \in [0, t_* + \epsilon]$ and $\text{Ray}(n')$ intersects $B \cap \Sigma^{sn}$ transversally. ρ is defined on U and $\rho(U) \subset B \cap \Sigma^{sn} \subset V \cap \Sigma^{sn}$. It follows that $U \subset A^{sn}$. The transversality of $\text{Ray}(n')$ and $B \cap \Sigma^{sn}$ and the smoothness of $B \cap \Sigma^{sn}$ imply that ρ is smooth. □

Lemma 2. *Suppose $n \in A^{hopf}$ and write $\rho(n) = \lambda_* = \lambda_0 + nt_* \in \Sigma_T^{hopf}$ and V for the neighborhood of λ_* such that $\Sigma_T^{hopf} \cap V$ is a smooth hypersurface containing λ_*. Then there is a neighborhood U of n with $U \subset A^{hopf}$, $\rho(U) \subset \Sigma^{hopf} \cap V$ and ρ smooth on U. It follows that A^{hopf} is open and that h is smooth on U.*

Proof. Since the Hopf bifurcation at λ_* satisfies the transversality conditions (2.4,2.5), there is a ball $B \subset V$ centered on λ_* of radius 2ϵ such that there is a Hopf bifurcation of x for $\lambda \in B$ iff $\lambda \in B \cap \Sigma^{hopf}$. Moreover, $B \cap \Sigma^{hopf}$ divides B into two connected components B^+, B^- with x exponentially stable for $\lambda \in B^-$ and unstable for $\lambda \in B^+$ (Sotomayor 1973, Sect. 3.4; Chow and Hale 1982, Sect. 9.5). For each $t \in [0, t_* - \epsilon]$, the definition of ρ ensures that x is exponentially stable for $\lambda = \lambda_0 + nt$ and $t \in [0, t_* - \epsilon]$. Indeed x is exponentially stable for λ in a ball centered on $\lambda_0 + nt$. Since $[0, t_* - \epsilon]$ is compact, there is an $\epsilon_1 > 0$ such that x is exponentially stable for $\lambda \in V_1 = \{ \lambda \mid |\lambda - (\lambda_0 + tn)| < \epsilon_1 \text{ for some } t \in [0, t_* - \epsilon] \}$. Now choose an open set $U \subset S^{m-1}$, $U \ni n$ such that for $n' \in U$, $\lambda_0 + n't \in V_1 \cup B$ for $t \in [0, t_* + \epsilon]$ and $\text{Ray}(n')$ intersects $B \cap \Sigma^{hopf}$ transversally. ρ is defined on U and $\rho(U) \subset B \cap \Sigma^{hopf} \subset V \cap \Sigma^{hopf}$. It follows that $U \subset A^{hopf}$. The transversality of $\text{Ray}(n')$ and $B \cap \Sigma^{hopf}$ and the smoothness of $B \cap \Sigma^{hopf}$ imply that ρ is smooth. □

Appendix 2: Regularity of the Direct Method at Solutions

We prove that the closest bifurcation systems (5.1), (5.4), and (6.3) are regular at solutions if $\lambda_* \in \Sigma_T$ and the curvature condition (3.1) is satisfied. The proof is more transparent when generalized; we assume extended system equations $G = 0$ for one-parameter variations which are regular at solutions and prove that their generalization to closest bifurcation equations $F = 0$ is regular at solutions. This general argument applies to the methods (5.1), (5.4), and (6.3) because the one-parameter extended system methods on which they are based are regular at solutions.

To define the one-parameter variation for the extended system equations $G = 0$, fix $n \in S^{m-1}$ and let $\lambda = \lambda_0 + nk$, $k \geq 0$. The extended system equations to be solved for (y, k) are

$$\begin{aligned} G(y, \lambda_0 + nk) &= 0, \\ G &: \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^{p+1}. \end{aligned} \tag{A2.1}$$

Note that the vector y contains the state vector x and eigenvectors of f_x. ($y = (x, w)$ for G corresponding to equations (5.1), $y = (x, \omega, v_1, v_2, w_1, w_2, u)$ for G corresponding to equations (5.4), and $y = (v, w)$ for G corresponding to equations (6.3).) If $\lambda_* \in \Sigma_T$ there is an open set $V \ni \lambda_*$ such that $\Sigma_T \cap V$ is a smooth hypersurface

and there is a smooth $u : \Sigma_T \cap V \to \mathbb{R}^p$ with $u(\lambda_*) = y_*$ and $G(u(\lambda_*'), \lambda_*') = 0$ for $\lambda_*' \in V$. We assert that if the bifurcation at λ_* satisfies the transversality conditions

$$N(\lambda_*)n \neq 0 \tag{A2.2}$$

and (2.2) in the case of saddle node and (2.5) in the case of Hopf and (6.1) in the case of transcritical or pitchfork, then G is regular at the solution (y_*, λ_*). This assertion is justified in the following three paragraphs.

In the saddle node case of equations (5.1.1)–(5.1.3), the regularity of G at solutions under conditions (A2.2) and (2.2) is proved in Moore and Spence (1980).

In the Hopf case of equations (5.4.1)–(5.4.10), it is sufficient to show that at a solution satisfying conditions (A2.2) and (2.4), $DG|_* z = 0$ implies that $z = 0$ where $z = (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \in \mathbb{R}^{6n+m+1}$. The results of Roose and Hlavecek (1985) prove that (5.4.1)–(5.4.5) are regular at a solution satisfying conditions (A2.2) and (2.4) and we can deduce from the structure of DG that $DG|_* z = 0$ implies that $z_1 = z_2 = z_3 = z_4 = z_5 = 0$. Now $DG|_* z = 0$ reduces to

$$(z_6^T f_x|_* - \omega z_7^T)\pi = 0, \tag{A2.3.1}$$

$$(z_7^T f_x|_* + \omega z_6^T)\pi = 0, \tag{A2.3.2}$$

$$z_6^T v_1 + z_7^T v_2 = 0, \tag{A2.3.3}$$

$$z_6^T v_2 - z_7^T v_1 = 0, \tag{A2.3.4}$$

$$f_x|_* z_8 = 0. \tag{A2.3.5}$$

$z_8 = 0$ follows from (A2.3.5) and the invertibility of $f_x|_*$. Equations (A2.3.1)–(A2.3.4) may be rewritten as

$$(z_6^T - iz_7^T)(f_x|_* - i\omega)\pi = 0, \tag{A2.4.1}$$

$$(z_6^T - iz_7^T)(v_1 + iv_2) = 0. \tag{A2.4.2}$$

Equation (A2.4.1) and $\ker\pi \cap \text{range}(f_x^T|_* - i\omega) = 0$ imply that $z_6^T - iz_7^T = \alpha(w_1 - iw_2)$ for $\alpha \in \mathbb{C}$. Substitution in (A2.4.2) gives $\alpha(w_1 - iw_2)(v_1 + iv_2) = 0$ and (5.4.8) implies that $\alpha = 0$. Hence $z_6 = z_7 = 0$ and $z = 0$ as required.

In the transcritical or pitchfork case of equations (6.3.1)–(6.3.4), it is sufficient to show that at a solution satisfying conditions (A2.2) and (6.1), $DG|_* z = 0$ implies that $z = 0$ where $z = (z_1, z_2, z_3) \in \mathbb{R}^{2n+1}$. $DG|_* z = 0$ may be written

$$f_x|_* z_1 + f_{x\lambda}(v, n)|_* z_3 = 0, \tag{A2.5.1}$$

$$2v^T z_1 = 0, \tag{A2.5.2}$$

$$z_2^T f_x|_* \pi + z_3(w f_{x\lambda}|_* n)\pi = 0, \tag{A2.5.3}$$

$$wz_1 + v^T z_2 = 0. \tag{A2.5.4}$$

Multiplying (A2.5.1) on the left by w yields $0 = wf_{x\lambda}(v, n)|_* z_3 = N(\lambda_*)nz_3$ so that (A2.2) implies that $z_3 = 0$. Hence (A.2.5.1) implies that $z_1 = \alpha v$ for some $\alpha \in \mathbb{R}$. Then (A2.5.2) implies that $\alpha = 0$ so that $z_1 = 0$. $z_3 = 0$ and (A2.5.3) imply that $z_2^T f_x|_* \in \ker\pi$. But $\ker\pi \cap \text{range} f_x^T|_* = 0$ so that $z_2^T f_x|_* = 0$ and $z_2^T = \beta w$ for some $\beta \in \mathbb{R}$. Finally (A2.5.4) and $v^T w = 1$ imply that $\beta = 0$ and $z_2 = 0$ so that $z = 0$ as required.

The generalization of the extended system equations $G = 0$ to closest bifurcation equations is

$$F(y, \lambda, k) = 0,$$
$$F : \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^{p+m+1}, \tag{A2.6}$$

where

$$F(y, \lambda, k) = \begin{cases} G(y, \lambda), \\ -k(\lambda - \lambda_0)^T + N(y, \lambda). \end{cases}$$

We assume a solution (y_*, λ_*, k_*) of (A2.6) so that $F(y_*, \lambda_*, k_*) = 0$, $\lambda_* \in \Sigma_T$ and the curvature condition (3.1) is satisfied. Consider the one-parameter variation $\lambda = \lambda_0 + N(\lambda_*)k$. The transversality condition (A2.2) is satisfied for this parameter variation so that $G(y, \lambda_0 + N(\lambda_*)k)$ is a regular function of y and k at (y_*, k_*). It follows that $G(y, \lambda)$ is a regular function of y and λ at (y_*, λ_*) and that $G^{-1}(0)$ is locally a smooth manifold H of dimension $m - 1$ containing (y_*, λ_*). The manifold $(u \times i)(\Sigma_T \cap V) = \{ (u(\lambda_*'), \lambda_*') \mid \lambda_*' \in \Sigma_T \cap V \}$ is contained in H since $G(u(\lambda_*'), \lambda_*') = 0$. In fact $(u \times i)(\Sigma_T \cap V) = H$ since the dimension of $(u \times i)(\Sigma_T \cap V)$ is $m - 1$. It follows that tangent vectors to H have the form $(u_{\lambda_*}(z_2), z_2)$ for $z_2 \in \mathrm{T}(\Sigma_T \cap V)$.

To prove that F is regular at solutions, we compute

$$DF = \begin{pmatrix} G_y & G_\lambda & 0 \\ N_y & N_\lambda - kI & \lambda - \lambda_0 \end{pmatrix}.$$

Suppose that $DF|_*(z_1, z_2, z_3)^T = 0$; to prove that F is regular at (y_*, λ_*, k_*) it suffices to prove that $z_1 = z_2 = z_3 = 0$. $(G_y z_1 + G_\lambda z_2)|_* = 0$ implies that $(z_1, z_2) \in \mathrm{T}H|_*$ and hence that $z_2 \in \mathrm{T}(\Sigma_T \cap V)|_*$ and $z_1 = u_{\lambda_*} z_2$. Equation (A2.6) and projection onto $\mathrm{T}(\Sigma_T \cap V)|_*$ along $N(y_*, \lambda_*)$ of

$$0 = (N_y z_1 + (N_\lambda - k_* I)z_2 + (\lambda_* - \lambda_0)z_3)|_* \tag{A2.7}$$

yield

$$(N_y u_{\lambda_*} z_2 + N_\lambda z_2)|_* = k_* z_2,$$

and since $z_2 \in \mathrm{T}(\Sigma_T \cap V)|_*$, $N_\lambda z_2 = N_{\lambda_*} z_2$, and

$$(N_y u_{\lambda_*} + N_{\lambda_*})|_* z_2 = k_* z_2,$$
$$D_{\lambda_*}(N(u(\lambda_*), \lambda_*))|_* z_2 = k_* z_2, \tag{A2.8}$$
$$\mathrm{II}|_*(z_2, z_2) = z_2^T D_{\lambda_*}(N(u(\lambda_*), \lambda_*))|_* z_2 = k_* z_2^T z_2.$$

Equation (A2.8) and the curvature condition (3.1) imply that $z_2 = 0$. Now $z_1 = u_\lambda z_2 = 0$ and (A2.7) gives $(\lambda_* - \lambda_0)z_3 = 0$ and $z_3 = 0$ since $\lambda_0 \neq \lambda_*$.

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