Strong Resonance Effects in Normal Form Analysis and Subsynchronous Resonance

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Abstract: Power system normal form analysis has developed coefficients and indices in modal coordinates to quantify nonlinear model interactions. The coefficients can become very large near a strong, nondiagonalizable resonance occurring in the power system linearization. Moreover, the changes in the coefficients when the power system equations are expressed in different coordinates or units show that the coefficients are not intrinsic. On a different topic, the paper suggests an explanation for the modal interaction that causes the subsynchronous resonance instability in power systems. The modal interaction is associated with a pair of strong resonances which arise as a perturbation of a weak resonance of complex eigenvalues. This idea is supported using results from the IEEE first benchmark subsynchronous resonance model and perturbation theory.

1 Introduction

Power systems are increasingly operated closer to their limits. Advances in communications, control, computing, signal processing and power electronics are enabling a more highly controlled power system. Stressed and more highly controlled power systems generally exhibit more nonlinear effects and dynamic modal interactions. To achieve a high performance power system that is controlled to reliably operate near its limits, dynamic modal interactions must be better understood so that they can be mitigated. This paper examines the effects and implications of strong modal resonance in power system normal form analysis and subsynchronous resonance. The effect of coordinate changes on normal form coefficients and indices is also studied.

1.1 Strong and weak resonance

The power system linearization and its modes vary as power system parameters change. In particular it is possible for two eigenvalues of the linearization to coincide and this is called a resonance, or a first order resonance. If the linearization is not diagonalizable at the resonance, the resonance is called a *strong* resonance [12]. Otherwise, if the linearization is diagonalizable at the resonance, the resonance is called a *weak* resonance. The resonance can occur with two eigenvalues coinciding on the real axis or with two complex conjugate pairs of eigenvalues coinciding in both

frequency and damping.

At a strong resonance, the two eigenvalues move together, briefly coincide and then split apart at right angles to the original direction of movement. Strong resonance of complex eigenvalues does not occur generically in systems with a single real parameter. Thus in practice the power system will not experience an exact strong resonance, but can pass close to such a resonance and the qualitative effects are similar: the eigenvalues move close together, quickly change direction by approximately a right angle and then move apart. The strong resonance and its implications for stability are known in mechanics [12, 13].

Dobson et al. [4] shows how passing near a complex strong resonance can cause oscillatory instability in power system models. The reader is referred to [4] for further explanation and analysis of strong resonance and a detailed review of previous indications that strong resonance may occur in interarea power system oscillations. [4] also shows that proximity to strong resonance is expected to occur generically in power systems and studies the effects of passing close to a single strong resonance. However, weak resonance, while generally much less generic than strong resonance, is also possible in power system models with special structure. Moreover, [4] raises the question of whether perturbations of weak resonance also occur in power systems.

1.2 Power system normal form analysis

Over the last decade normal form analysis has been developed to investigate and quantify nonlinear interactions between power system modes [9, 14, 15, 17]. Applications include control system design [9, 7] and predicting interarea separation [14, 15]. The method starts with power system differential equations which are expanded in a Taylor series about a stable equilibrium. The differential equations are then transformed so that the linearization is diagonalized and the equations are written in modal coordinates. The coefficients of the quadratic terms in the modal coordinates are the quantities C^i_{jk}, where i, j and k are indices ranging from 1 to n, the number of state variables. Then the equations are nonlinearly transformed to be linear up to second order. The sizes of second order terms $h2^i_{jk}$ in the nonlinear transformation are used to quantify nonlinear interactions between the modes. The $h2^i_{jk}$ are calculated from the C^i_{jk}

using

$$h2^i_{jk} = \frac{C^i_{jk}}{\lambda_j + \lambda_k - \lambda_i} \qquad (1)$$

where λ_1, λ_2, ... , λ_n are the eigenvalues of the power system linearization.

A *second order resonance* occurs when two eigenvalues sum to equal a third eigenvalue. The formula (1) shows that the $h2^i_{jk}$ become large near a second order resonance in which $\lambda_j + \lambda_k - \lambda_i$ vanishes. Nonlinear mode couplings become large near second order resonance. Note that, in contrast, the strong and weak resonances discussed above are first order resonances which occur within the system linearization.

1.3 Subsynchronous resonance

In regions with long transmission lines, such as the western United States, it can be desirable to compensate the lines with series capacitors to increase the maximum power transfer through the lines. However, series capacitors can introduce resonance problems. Subsynchronous resonance is an electromechanical power system instability in which electrical modes of transmission lines compensated with series capacitors interact with torsional modes of generator shafts [5, 1, 11]. This instability can break generator shafts and must be studied and prevented when series compensation is used.

1.4 Objectives and organization of paper

After reviewing the power system normal form analysis procedure in section 2, the subsequent sections have the following objectives:

- Section 3 examines normal form analysis near strong resonance. Some of the coefficients used to quantify nonlinear interactions become very large near strong resonance.

- Section 4 studies how coefficients used to quantify nonlinear interactions in normal form analysis change when different coordinates or units are used in the power system model.

- Section 5 explores the anatomy of the subsynchronous resonance instability using concepts of strong and weak resonance. Perturbations of a weak resonance of complex eigenvalues are analyzed and illustrated. The results suggest that subsynchronous resonance is caused by a pair of strong resonances which are a perturbation of a weak resonance.

Sections 3, 4 and 5 may be read independently. Section 6 summarizes the conclusions.

2 Normal form analysis procedure

This section reviews the mechanics of the normal form calculation of Lin, Vittal, Kliemann, and Fouad [9] and Starrett and Fouad [14]. We follow much of the notation of [9] for convenience.

The power system dynamics are linearized about a stable equilibrium and expanded in a Taylor series to obtain

$$\dot{x}_i = \sum_{j=1}^{n} A_{ij} x_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} H^i_{jk} x_j x_k + h.o.t. \qquad (2)$$

where $X = (x_1, x_2, ..., x_n)^t$ is the system state defined relative to the stable equilibrium, A is the Jacobian matrix and the Hessian H defines the coefficients of quadratic terms. When transients are considered, the initial condition is written as $X_0 = (x_{10}, x_{20}, ..., x_{n0})^t$.

The Jacobian matrix A is assumed to be diagonalizable. Equation (2) is transformed to modal coordinates $Y = (y_1, y_2, ..., y_n)^t$ which diagonalize the matrix A. In particular, U is defined to be a matrix whose columns are the right eigenvectors of A. An important detail [9] is that the right eigenvectors (columns of U) are normalized so that they have length 1. (An assumption about the normalization of U is needed to unambiguously define the C^i_{jk} below.) Then

$$X = UY \qquad (3)$$

is a transformation to modal Y coordinates. Transforming (2) to the modal Y coordinates yields

$$\dot{y}_i = \lambda_i y_i + \sum_{j=1}^{n} \sum_{k=1}^{n} C^i_{jk} y_j y_k + h.o.t. \qquad (4)$$

where λ_1, λ_2, ... , λ_n are the eigenvalues of A.

The C^i_{jk} are the quadratic coefficients in the Y coordinates and are given by

$$C^i_{jk} = \frac{1}{2} \sum_{r=1}^{n} \sum_{\ell=1}^{n} \sum_{m=1}^{n} (U^{-1})_{ir} H^r_{\ell m} U_{\ell j} U_{mk} \qquad (5)$$

The $h2$ coefficients are then defined by (1).

Normal form theory states that the local nonlinear transformation from Y to Z variables given by

$$y_i = z_i + \sum_{j=1}^{n} \sum_{k=1}^{n} h2^i_{jk} z_j z_k \qquad (6)$$

linearizes the differential equations to second order so that

$$\dot{z}_i = \lambda_i z_i + \text{third order terms} \qquad (7)$$

An initial condition X_0 for a transient becomes $Y_0 = U^{-1} X_0$ in the Y coordinates and Z_0 in the Z coordinates

where $Z_0 = (z_{10}, z_{20}, ..., z_{n0})^t$ satisfies

$$y_{i0} = z_{i0} + \sum_{j=1}^{n} \sum_{k=1}^{n} h2_{jk}^i z_{j0} z_{k0} \tag{8}$$

Several measures of the nonlinearity in Y coordinates were proposed in [15] based on the $h2$ coefficients and the initial condition of a transient. An interaction coefficient is defined as

$$h2_{jk}^i z_{j0} z_{k0} \tag{9}$$

to quantify the effect of second order terms on the transient solution. The nonlinear interaction index $I1$ for mode i is defined as

$$I1(i) = |y_{i0} - z_{i0} + h2_{j^*k^*}^i z_{j^*0} z_{k^*0}| \tag{10}$$

where the choice of $j = j^*$ and $k = k^*$ maximizes the interaction coefficient size $|h2_{jk}^i z_{j0} z_{k0}|$. The nonlinearity index $I2$ for mode i is defined as

$$I2(i) = \frac{h2_{j^*k^*}^i z_{j^*0} z_{k^*0}}{z_{i0}} \tag{11}$$

We do not address the generalized participation factor analysis of [14, 18] in this paper.

3 Normal form analysis near strong resonance

In the context of the normal form analysis, strong resonance is a first order resonance occurring within the power system linearization. We present examples showing that the coefficients and indices C_{jk}^i, $h2_{jk}^i$, $h2_{jk}^i z_{j0} z_{k0}$, $I1$, $I2$ become arbitrarily large near strong resonance despite the very small amounts of nonlinearity in the system expressed in its original coordinates.

3.1 Real strong resonance example

This subsection gives a two dimensional example in which the system is near to a strong resonance of two eigenvalues on the real axis. (A version of this example appeared in [3, appendix B] with correct conclusions but an incorrect constant factor for the U matrix.) Consider

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= \mu x_1 - x_2 + \tfrac{1}{2}\epsilon x_1^2 \end{aligned} \tag{12}$$

so that $A = \begin{pmatrix} -1 & 1 \\ \mu & -1 \end{pmatrix}$ and all $H_{jk}^i = 0$ except that $H_{11}^2 = \epsilon$. ϵ and μ are small real constants. The eigenvalues of A are $-1 \pm \sqrt{\mu}$ and these coincide in a strong resonance at -1 when $\mu = 0$. We assume $\mu \neq 0$ in order to diagonalize A and compute

$$U = \frac{1}{\sqrt{1+\mu}} \begin{pmatrix} 1 & 1 \\ \sqrt{\mu} & -\sqrt{\mu} \end{pmatrix} \tag{13}$$

$$U^{-1} = \frac{\sqrt{1+\mu}}{2} \begin{pmatrix} 1 & \frac{1}{\sqrt{\mu}} \\ 1 & \frac{-1}{\sqrt{\mu}} \end{pmatrix} \tag{14}$$

Then (12) in Y coordinates becomes

$$\begin{aligned} \dot{y}_1 &= (-1 + \sqrt{\mu})y_1 + \frac{\epsilon}{4\sqrt{\mu}\sqrt{1+\mu}}(y_1^2 + 2y_1 y_2 + y_2^2) \\ \dot{y}_2 &= (-1 - \sqrt{\mu})y_2 - \frac{\epsilon}{4\sqrt{\mu}\sqrt{1+\mu}}(y_1^2 + 2y_1 y_2 + y_2^2) \end{aligned}$$

and

$$C_{jk}^i = \frac{\epsilon(-1)^{i+1}}{4\sqrt{\mu}\sqrt{1+\mu}} \tag{15}$$

$$h2^1 = \frac{\epsilon}{4\sqrt{\mu}\sqrt{1+\mu}} \begin{pmatrix} \frac{1}{-1+\sqrt{\mu}} & \frac{1}{-1-\sqrt{\mu}} \\ \frac{1}{-1-\sqrt{\mu}} & \frac{1}{-1-3\sqrt{\mu}} \end{pmatrix} \tag{16}$$

$$h2^2 = \frac{\epsilon}{4\sqrt{\mu}\sqrt{1+\mu}} \begin{pmatrix} \frac{-1}{-1+3\sqrt{\mu}} & \frac{-1}{-1+\sqrt{\mu}} \\ \frac{-1}{-1+\sqrt{\mu}} & \frac{-1}{-1-\sqrt{\mu}} \end{pmatrix} \tag{17}$$

Note the factor $\frac{\epsilon}{\sqrt{\mu}}$ in (15) and (16,17).

The appendix considers the behavior of the initial condition Z_0 near strong resonance and proves that one can find values of μ tending to zero such that one or more of the interaction coefficients $h2_{jk}^i z_{j0} z_{k0}$ becomes arbitrarily large. Indices $I1$ and $I2$ become arbitrarily large in the same way.

3.2 Complex strong resonance example

This subsection gives a four dimensional example in which the system is near to a strong resonance of two complex conjugate pairs of eigenvalues.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 & 0 \\ \mu & -1 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & \mu & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2}\epsilon x_1^2 \\ 0 \\ 0 \end{pmatrix} \tag{18}$$

All $H_{jk}^i = 0$ except that $H_{11}^2 = \epsilon$. ϵ and μ are small real constants. The eigenvalues of the matrix in (18) are $\lambda_1 = -1 + i + \sqrt{\mu}$, $\lambda_2 = -1 + i - \sqrt{\mu}$ and their complex conjugates λ_1^*, λ_2^*. The eigenvalues λ_1 and λ_2 coincide in a strong resonance at $-1 + i$ when $\mu = 0$. (λ_1^* and λ_2^* mirror this behavior below the real line by coinciding in a strong resonance at $-1 - i$ when $\mu = 0$.) We assume $\mu \neq 0$ in order to diagonalize the matrix in (18) and compute

$$U = \frac{1}{\sqrt{2}\sqrt{1+\mu}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \sqrt{\mu} & -\sqrt{\mu} & \sqrt{\mu} & -\sqrt{\mu} \\ i & i & -i & -i \\ i\sqrt{\mu} & -i\sqrt{\mu} & -i\sqrt{\mu} & i\sqrt{\mu} \end{pmatrix}$$

$$U^{-1} = \frac{\sqrt{1+\mu}}{2\sqrt{2}} \begin{pmatrix} 1 & \frac{1}{\sqrt{\mu}} & -i & \frac{-i}{\sqrt{\mu}} \\ 1 & \frac{-1}{\sqrt{\mu}} & -i & \frac{i}{\sqrt{\mu}} \\ 1 & \frac{1}{\sqrt{\mu}} & i & \frac{i}{\sqrt{\mu}} \\ 1 & \frac{-1}{\sqrt{\mu}} & i & \frac{-i}{\sqrt{\mu}} \end{pmatrix}$$

Then (18) in Y coordinates becomes

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{pmatrix} = \begin{pmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \\ \lambda_1^* y_3 \\ \lambda_2^* y_4 \end{pmatrix} + \frac{\epsilon(y_1 + y_2 + y_3 + y_4)^2}{8\sqrt{2}\sqrt{\mu}\sqrt{1+\mu}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad (19)$$

and

$$C_{jk}^i = \frac{\epsilon(-1)^{i+1}}{8\sqrt{2}\sqrt{\mu}\sqrt{1+\mu}} \quad (20)$$

$$h2_{jk}^i = \frac{\epsilon(-1)^{i+1}}{8(\lambda_j + \lambda_k - \lambda_i)\sqrt{2}\sqrt{\mu}\sqrt{1+\mu}} \quad (21)$$

Note the factor $\frac{\epsilon}{\sqrt{\mu}}$ in (20) and (21).

Similarly to the real resonance case in the appendix, one can find values of μ tending to zero such that an interaction coefficient $h2_{jk}^i z_{j0} z_{k0}$ and the indices $I1$ and $I2$ become arbitrarily large.

3.3 Discussion

Both examples show that near strong resonance, the C_{jk}^i and h_{jk}^i coefficients scale as $\frac{\epsilon}{\sqrt{\mu}}$. Here ϵ controls the amount of nonlinearity in the original coordinates ($\epsilon = 0$ gives no nonlinearity) and μ controls the proximity to strong resonance ($\mu = 0$ gives a strong resonance). No matter how small ϵ is, one can choose a smaller μ so that the coefficients C_{jk}^i and h_{jk}^i have very large magnitudes. Indeed C_{jk}^i and h_{jk}^i tend to infinity as the strong resonance is approached. Moreover, for both examples, an interaction coefficient $h2_{jk}^i z_{j0} z_{k0}$ and the indices $I1$ and $I2$ become arbitrarily large for values of μ which tend to zero.

That is, no matter how small is the quadratic nonlinearity in the equations in the original X coordinates, one can, by moving close to the strong resonance, make the coefficients and indices C_{jk}^i, h_{jk}^i, $h2_{jk}^i z_{j0} z_{k0}$, $I1$, $I2$ arbitrarily large. Therefore caution is needed in interpreting these coefficients and indices near strong resonance. Large values of these coefficients and indices correctly reflect the high nonlinearity of the system in the modal Y coordinates, but do not necessarily imply that the system has significant nonlinearity in the original X coordinates. The coordinate change to modal Y coordinates requires increasing distortion as strong resonance is approached and this distortion can greatly amplify nonlinearities. These results show that, as well as quantifying second order nonlinear modal interactions, normal form analysis also detects strong resonance effects (first order linear modal interactions) which are not necessarily related to significant system nonlinearity in the original X coordinates.

The occurrence of strong resonance also has some effect on the procedure to determine second order resonance condi-

tions in [17, section IVB]. This procedure requires eigenvalues to vary smoothly enough with respect to parameters so that they can be approximated by a quadratic polynomial in the parameters. However, this is not possible at strong resonance, where eigenvalues are not differentiable with respect to parameters. At strong resonance the eigenvalues vary as fractional powers of a parameter despite the smoothness or analyticity of the power system equations. (Note the dependence of the eigenvalues on $\sqrt{\mu}$ in subsections 3.1 and 3.2.) One also expects accuracy problems when trying to approximate eigenvalue movement with quadratic polynomials near a strong resonance.

4 Dependence of coefficients and indices on coordinate systems and units

The coefficients and indices C_{jk}^i, $h2_{jk}^i$, $h2_{jk}^i z_{j0} z_{k0}$, $I1$, $I2$ depend on the coordinate system used to express the power system differential equations. In particular, the values of these coefficients and indices vary if the state variables are expressed in different units.

The essence of the problem can be seen in a simple example. Consider the following one dimensional differential equation in which the state variable x is measured in MW.

$$\dot{x} = -x + x^2 \quad (22)$$

If we change the state variable to be $\bar{x}$ where $\bar{x}$ is measured in kW, then

$$\bar{x} = 1000\,x \quad (23)$$

and (22) becomes

$$\dot{\bar{x}} = -\bar{x} + 0.001\bar{x}^2 \quad (24)$$

For (22) it is apparent that $C_{11}^1 = 1$. C_{11}^1 is intended to quantify the nonlinear interaction of mode 1 (the only mode) of system (22) with itself. But in (24), which represents the same physical system in different units, we have $\bar{C}_{11}^1 = 0.001$. It is clear that the size of the nonlinearity as measured by the C_{11}^1 coefficient varies according to the units chosen to express the differential equation. Note that both x and $\bar{x}$ are modal coordinates; there is no unique choice of modal coordinates.

There are also changes in $h2_{11}^1$ and the interaction coefficient $h2_{11}^1 z_0^2$ when the units are changed. For (22), $h2_{11}^1 = -1$, whereas for (24), $\bar{h}2_{11}^1 = -0.001$. For (22), solving (8) and (3) gives $z_0 = \frac{1}{2} - \sqrt{\frac{1}{4} - x_0}$ and

$$h2_{11}^1 z_0^2 = x_0 - \frac{1}{2} + \sqrt{\frac{1}{4} - x_0} \quad (25)$$

whereas for (24), $\bar{z}_0 = 500 - \sqrt{250000 - 1000\bar{x}_0}$ and

$$\bar{h}2_{11}^1 \bar{z}_0^2 = \bar{x}_0 - 500 + \sqrt{250000 - 1000\bar{x}_0} = 1000\,h2_{11}^1 z_0^2 \quad (26)$$

4.1 Linear transformation of coordinates

We transform the differential equations from the original X coordinates to $\bar{X}$ coordinates and then compute the coefficients and indices $\bar{C}^i_{jk}$, $\bar{h2}^i_{jk}$, $\bar{h2}^i_{jk}z_{j0}z_{k0}$, $\bar{I}1$, $\bar{I}2$. In this way we find out how these coefficients and indices depend on the coordinate system from which they are computed.

Suppose that the X and $\bar{X}$ coordinates are related by an invertible linear transformation T according to

$$X = T\bar{X} \tag{27}$$

Then the differential equations in $\bar{X}$ coordinates are

$$\dot{\bar{x}}_i = \sum_{j=1}^n \bar{A}_{ij}\bar{x}_j + \frac{1}{2}\sum_{j=1}^n\sum_{k=1}^n \bar{H}^i_{jk}\bar{x}_j\bar{x}_k + h.o.t.$$

where

$$\bar{A} = T^{-1}AT \tag{28}$$

$$\bar{H}^i_{jk} = \sum_{r=1}^n\sum_{\ell=1}^n\sum_{m=1}^n (T^{-1})_{ir}H^r_{\ell m}T_{\ell j}T_{mk} \tag{29}$$

Right eigenvectors in the $\bar{X}$ coordinates are given by the columns of $T^{-1}U$, but in general these columns will not be of length 1. Therefore we define

$$\bar{U} = T^{-1}UN \tag{30}$$

where N is a diagonal matrix chosen so that $\bar{U} = T^{-1}UN$ is normalized to have columns of length 1. A formula for the diagonal elements of N is

$$N_{ii} = \left[(U^t(T^{-1})^tT^{-1}U)_{ii} \right]^{-\frac{1}{2}} \tag{31}$$

(An arbitrary choice about the phase and sign of elements of N can be made if it is agreed that only the magnitude of coefficients is of interest.) Now

$$\bar{C}^i_{jk} = \frac{1}{2}\sum_{r=1}^n\sum_{\ell=1}^n\sum_{m=1}^n (\bar{U}^{-1})_{ir}\bar{H}^r_{\ell m}\bar{U}_{\ell j}\bar{U}_{mk}$$

Substituting from (29), (30), using (5) and simplifying yields

$$\bar{C}^i_{jk} = \sum_{r=1}^n\sum_{\ell=1}^n\sum_{m=1}^n (N^{-1})_{ir}C^r_{\ell m}N_{\ell j}N_{mk} \tag{32}$$

and, since N is diagonal, (32) becomes

$$\bar{C}^i_{jk} = N_{ii}^{-1}C^i_{jk}N_{jj}N_{kk} \tag{33}$$

Since the eigenvalues are coordinate independent, formula (1) shows that the $h2^i_{jk}$ transform in the same way as the C^i_{jk}:

$$\bar{h}^i_{jk} = N_{ii}^{-1}h^i_{jk}N_{jj}N_{kk} \tag{34}$$

Equations (3), (30), (27) and $\bar{X} = \bar{U}\bar{Y}$ imply that

$$\bar{Y} = N^{-1}Y \tag{35}$$

If Z and Y satisfy (6), then it can be shown using (34) and (35) that $N^{-1}Z$ and $\bar{Y}$ satisfy

$$\bar{y}_i = \bar{z}_i + \sum_{j=1}^n\sum_{k=1}^n \bar{h2}^i_{jk}\bar{z}_j\bar{z}_k \tag{36}$$

Therefore

$$\bar{Z} = N^{-1}Z \tag{37}$$

Equations (37) and (34) imply that the interaction coefficient transforms as

$$\bar{h2}^i_{jk}\bar{z}_{j0}\bar{z}_{k0} = N_{ii}^{-1}h2^i_{jk}z_{j0}z_{k0} \tag{38}$$

Equations (35), (37) and (38) seem at first glance to imply that the indices $I1$ and $I2$ transform as

$$\bar{I}1(i) = N_{ii}^{-1}I1(i) \tag{39}$$
$$\bar{I}2(i) = I2(i) \tag{40}$$

but (39) and (40) only apply in the cases in which the coordinate change does not alter the choice of j^* and k^* determining the interaction coefficient of maximum size $h2^i_{j^*k^*}z_{j^*0}z_{k^*0}$ in the formulas (10) and (11).

However, we can suggest modifying the interaction coefficient to

$$\frac{h2^i_{jk}z_{j0}z_{k0}}{z_{i0}} \tag{41}$$

and modifying the index $I2$ to

$$I3(i) = \max_{j,k} \left| \frac{h2^i_{jk}z_{j0}z_{k0}}{z_{i0}} \right| \tag{42}$$

The new coefficient (41) and index (42) are constants invariant under coordinate change and are intrinsic to the system.

We consider special cases of the transformation T in which the formula (31) for N becomes more simple:

1. Diagonal T can be interpreted as a transformation which changes the units in which the system states are measured. If T is diagonal, then

$$N_{ii} = \left[\sum_{j=1}^n U^t_{ij}U_{ji}T_{jj}^{-2} \right]^{-\frac{1}{2}} \tag{43}$$

Formula (43) generally simplifies to $N = T$ only when T is a multiple of the identity matrix.

2. If T is orthonormal so that $T^tT = I$, then $N = I$ and all the coefficients and indices are invariant under T.

4.2 Discussion

The transformation formulas (33), (34) and (38) show how the coefficients C^i_{jk}, $h2^i_{jk}$ and $h2^i_{jk}z_{j0}z_{k0}$ change when the coordinates assumed for the differential equations are changed by a linear transformation. Since the transformation formulas depend on the matrix N which normalizes the columns of U, these coefficients are neither independent of the coordinates assumed for the differential equations, nor do they transform as a tensor with respect to those coordinates. Thus the coefficients C^i_{jk}, $h2^i_{jk}$ and $h2^i_{jk}z_{j0}z_{k0}$ are not intrinsic to the system under study. This can be problematic for quantifying nonlinear interactions with these coefficients. For example, the ranking of these coefficients by size can change when different coordinates are used.

For a general linear transformation of the coordinates assumed for the differential equations, the coefficients C^i_{jk}, $h2^i_{jk}$ and $h2^i_{jk}z_{j0}z_{k0}$ transform in way that depends in a complicated fashion on the linear transformation. A similarly complicated dependence would also arise in the case of a nonlinear coordinate transformation such as changing phasors from polar coordinates to rectangular coordinates.

A diagonal transformation of coordinates T (case 1 in subsection 4.1) is an important special case because it represents an independent rescaling of each of the state variables, or, equivalently, a change in units. In this special case, the N matrix is given by (43) and the coefficients have the nontrivial transformations (33), (34) and (38). In power system models, there is a mixed set of units because the state variables include different physical quantities and these transformations show how the coefficients change when, for example, angles are measured in degrees instead of radians, or when per unit scaling is introduced or removed for some of the states.

One perspective to help explain the transformations of the coefficients C^i_{jk}, $h2^i_{jk}$ and $h2^i_{jk}z_{j0}z_{k0}$ is as follows: The idea of these coefficients is to transform the system to a standard coordinate system, namely modal coordinates that diagonalize the Jacobian, and to measure the nonlinear interactions in that particular coordinate system. However, there are no unique modal coordinates; any given set of modal coordinates can be independently rescaled to yield another set of modal coordinates. (There appears, at this stage of inquiry, to be no intrinsic way to select one of these sets of modal coordinates as canonical.) The scaling of the modal coordinates is chosen by the scaling of the eigenvectors (columns of U). Different coordinate systems for the original differential equations yield different choices of scalings for the modal coordinates (see (35)) and hence different values of the coefficients.

The $I1$ and $I2$ indices depend on which interaction coefficient has maximum size. Since the interaction coefficient of maximum size can vary with the coordinate system, the $I1$ and $I2$ indices can vary in a nonstandard way when the coordinate system changes. That is, the $I1$ and $I2$ indices depend on a ranking by size of the interaction coefficients which is not invariant. (In the absence of this effect, the $I2$ index would be a constant independent of the coordinate system (see (40)), but the $I1$ index would vary with the coordinate system (see (39)).)

The most recent work applying normal form analysis [18] mentions the difficulty of quantifying nonlinearity with the $I1$ and $I2$ indices in comparing cases at different operating conditions: "However, using these indices, it was difficult to compare the nonlinearity quantitatively because of the scaling of the eigenvectors." This statement is consistent with our results.

We suggest a new coefficient (41) and index (42) which are constants invariant under coordinate change. The new coefficient and index are similar to the noninvariant measures proposed in [15] and their meaning and usefulness should be evaluated in future work.

5 Subsynchronous resonance

We suggest a detailed explanation of the eigenvalue movements in the subsynchronous resonance instability using concepts of strong and weak resonance. The results are illustrated using calculations on the IEEE first benchmark model [6, 11, 1] and an ideal coupled oscillator system and are supported by a theory of perturbations of parameterized matrices.

In the IEEE first benchmark model, the frequency of the subsynchronous electrical mode can be changed by varying the amount of capacitive series compensation. It is well known that as the frequency of the subsynchronous electrical mode in the generator rotor passes near the frequency of a torsional mode, the torsional mode rapidly destabilizes and then stabilizes. We propose that that the destabilization and stabilization are caused by passing close to a pair of strong resonances. Passing close to the first strong resonance causes the destabilization and then passing close to the second strong resonance causes the stabilization.

How does the pair of strong resonances arise? It turns out that the system is close to a weak resonance, and perturbing a weak resonance generally causes a pair of strong resonances to arise.

5.1 Related work by Bowler, Padiyar and Seyranian

In a classic paper [2, example 3], Bowler, Ewart, and Concordia analyze eigenvalue movements in a pair of coupled RLC oscillators to give physical insight into subsynchronous resonance. Our perspective is that the eigenvalue movements in [2, figure 6b] show a pair of strong resonances and

subsequent figures show perturbations of the strong resonances. Section 5.2 shows similar results.

Padiyar [11, section 1.5] associates subsynchronous resonance with the "mode coupling" described by Van Ness [10] in the context of interarea oscillations. It is suggested in [4] that the similar mode coupling described by Van Ness in [16] is a strong resonance phenomenon. Padiyar [11] also describes the mode coupling in subsynchronous resonance in a way consistent with the close eigenvalues and nearly aligned eigenvectors found near strong resonance: "The eigenvalues that are close can result in mode coupling (although not always). Actually, the mode coupling arises from the similarity between eigenvectors." We can read [11] as strengthening the notion that strong resonance effects occur in subsynchronous resonance.

Seyranian [12, section IIIC] considers the perturbation of a weak resonance of two real eigenvalues into two strong resonances for the vibrational linear system

$$M\ddot{x} + \epsilon B\dot{x} + F = 0 \qquad (44)$$

M is positive definite and symmetric and F is negative definite and symmetric. When $\epsilon = 0$, the system is self adjoint and the system has real eigenvalues which can coincide in a weak resonance.

Then B is assumed to be antisymmetric and the weak resonance is perturbed by considering small, nonzero ϵ. This models a small gyroscopic force. The weak resonance perturbs to a pair of strong resonances which cause a "bubble" in which the eigenvalues move off the real axis between the two strong resonances. This case, in which the perturbed eigenvalues pass exactly through the pair of strong resonances, is a perturbation of a weak resonance of real eigenvalues. For subsynchronous resonance we consider below perturbation of a weak resonance of complex eigenvalues.

5.2   Two coupled linear oscillators

Before examining subsynchronous resonance, we look at a much simpler case which is easier to analyze. Following [2, example 3], we consider the two linear oscillators

$$\ddot{x} + \dot{x} + \omega^2 x = b\dot{y} \qquad (45)$$
$$\ddot{y} + \delta\dot{y} + 100y = a\dot{x} \qquad (46)$$

The oscillators are mutually coupled by the terms $b\dot{y}$, $a\dot{x}$ on the right hand sides of (45), (46). In the absence of this coupling ($a = b = 0$), the first oscillator has damping 0.5 and natural frequency ω, and the second oscillator has damping $\delta/2$ and natural frequency 10. The oscillators can be written in state space form as

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega^2 & -1 & 0 & b \\ 0 & 0 & 0 & 1 \\ 0 & a & -100 & -\delta \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{pmatrix} \qquad (47)$$

We are interested in the resonances and interactions of eigenvalues as the first oscillator natural frequency ω is varied to pass near the second oscillator natural frequency 10 for various values of the other parameters. The eigenvalues are the roots of a quartic characteristic equation which we explicitly but messily solve using computer algebra. It is convenient to focus on the two eigenvalues λ_1 and λ_2 of positive frequency; the complex conjugate eigenvalues with negative frequency move similarly below the real axis.

First suppose that the oscillators are completely uncoupled so that $a = b = 0$ and that the second oscillator has damping $\delta = 0$. Then the first oscillator has eigenvalue $\lambda_1 = -0.5 + \sqrt{0.25 - \omega^2}$ and the second oscillator has a fixed eigenvalue $\lambda_2 = 10j$. The movement of λ_1 as ω is decreased from 12 to 7 is shown in Fig. 1. There is no interaction between the eigenvalues.

If the damping of the second oscillator is changed to $\delta = 1$, its eigenvalue changes to $\lambda_2 = -0.5 + 9.99j$, but the movement of λ_1 is unchanged and λ_1 passes through $-0.5 + 9.99j$ when $\omega = 10$ as shown in Fig. 2. Thus for $\delta = 1$ and no coupling between the oscillators, the two eigenvalues coincide in a weak resonance at $\omega = 10$.
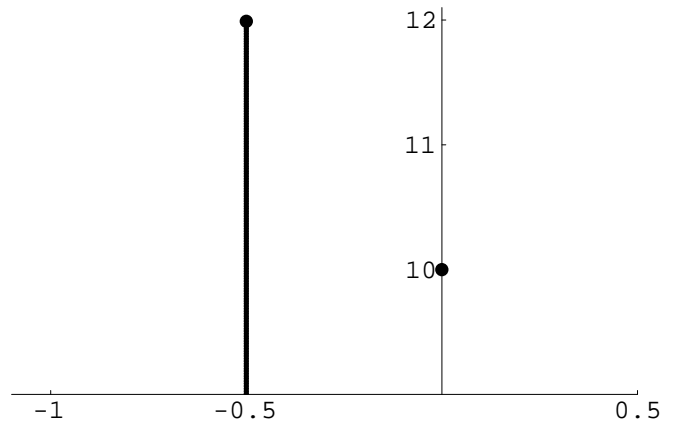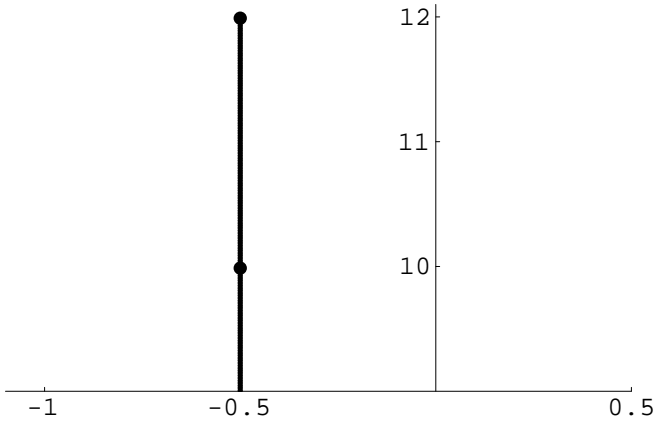

Fig. 1: Linear oscillator eigenvalues as ω is decreased from 12 to 7; $a = b = 0$ (no coupling) and $\delta = 0$. Larger dot shows initial position of eigenvalues and movement below frequency 9 is not shown.

Now include the mutual coupling between the oscillators so that $a = b = 1$. When the damping δ is zero, the eigenvalues interact as shown in Fig. 3. Fig. 3 is a perturbation of Fig. 1 caused by adding the coupling.

To explain the interaction in Fig. 3 in terms of strong resonance effects, we set $\delta = 1$ and start by perturbing the weak resonance in Fig. 2. Adding the mutual coupling $a = b = 1$ between the oscillators perturbs the weak resonance in Fig. 2 to Fig. 4. In Fig. 4, we claim that λ_1 starts at the top, moves downward, then interacts with λ_2 by passing near a strong resonance, moves to the right, moves to the left and then interacts again with λ_2 by passing near another strong resonance. The changes of direction

Fig. 2: Linear oscillator eigenvalues as ω is decreased from 12 to 7; $a = b = 0$ (no coupling) and $\delta = 1$. Larger dot shows initial position of eigenvalues and movement below frequency 9 is not shown.
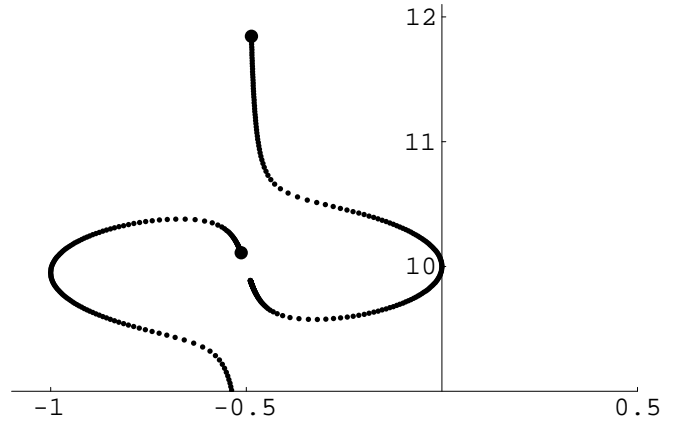

Fig. 4: Linear oscillator eigenvalues near two strong resonances; $a = b = 1$ (mutual coupling) and $\delta = 1.0$.
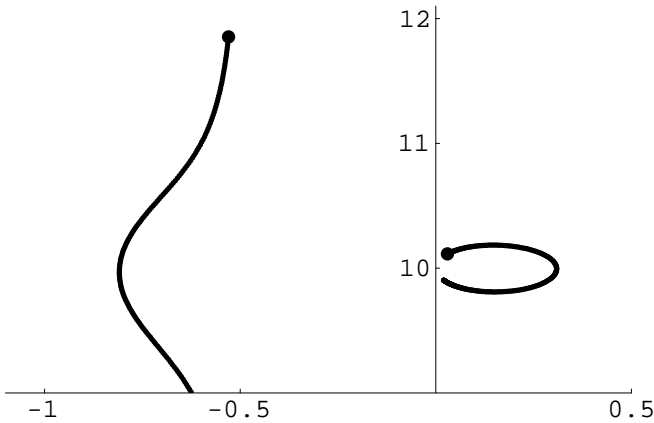

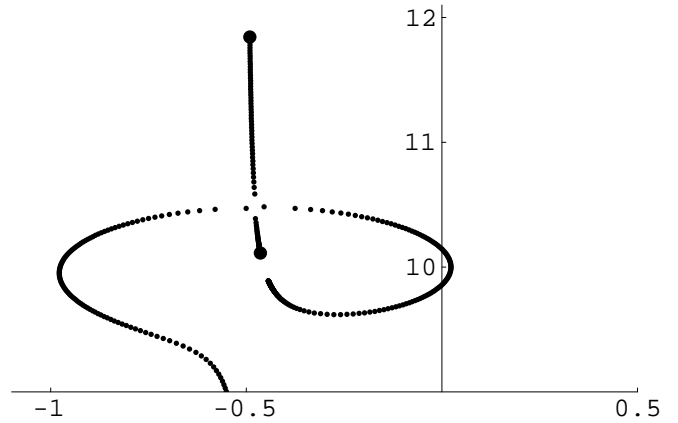Fig. 3: Linear oscillator eigenvalues as ω is decreased from 12 to 7; $a = b = 1$ (mutual coupling) and $\delta = 0$.

Fig. 5: Linear oscillator eigenvalues showing exact strong resonance in the first interaction; $a = b = 1$ and $\delta = 0.91$.

of eigenvalues when the eigenvalues are close are characteristic of movement near strong resonance [4].

Indeed, we can adjust the damping δ near $\delta = 1$ to find the two strong resonances. $\delta = 0.91$ makes the first interaction between the eigenvalues an exact strong resonance as shown in Fig. 5. $\delta = 1.11$ makes the second interaction between the eigenvalues an exact strong resonance as shown in Fig. 6. This confirms the proximity to two strong resonances in Fig. 4.

Now we reduce the damping from $\delta = 1.0$ in Fig. 4 to $\delta = 0$ in Fig. 3 and note that the effect of the two nearby strong resonances is still perceptible in the eigenvalue movement. To show more clearly the passage from Fig. 4 to Fig. 3 we show an intermediate case with $\delta = 0.89$ in Fig. 7. Note that the eigenvalue which becomes unstable during the resonance changes at the damping $\delta = 0.91$ while passing through the first strong resonance in Fig. 5; this effect is expected [4, 13].

5.3 Subsynchronous resonance modal interactions

The first benchmark model was created by the IEEE subsynchronous resonance Task Force [6] as a standard test model to facilitate analysis and comparison of calculations of subsynchronous resonance. The turbine-generator shaft is modeled as 6 masses with torsional springs. There are five torsional modes TM1 through TM5 for the mechanical system with respective natural frequencies 100, 127, 160, 203, 298 radian/s.

The equations and parameters of the IEEE first benchmark model are taken from [11]. The no load case is assumed. The mechanical damping is initially assumed to be zero. The subsynchronous mode frequency is varied by changing the series capacitance X_c from 0.12 to 0.55 pu. This corresponds to varying the transmission line compensation from 24% to 110%. (It is convenient to extend the compensation to the artificially high value of 110% so that the more dramatic eigenvalue movements of the TM1 resonance can be seen in the results.)
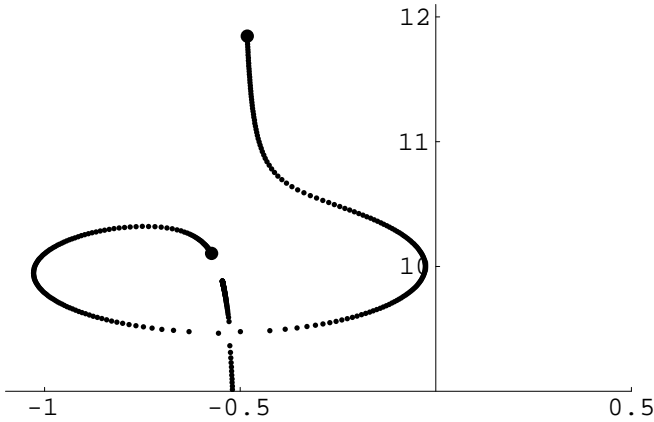
Fig. 6: Linear oscillator eigenvalues showing exact strong resonance in the second interaction; $a = b = 1$ and $\delta = 1.11$.
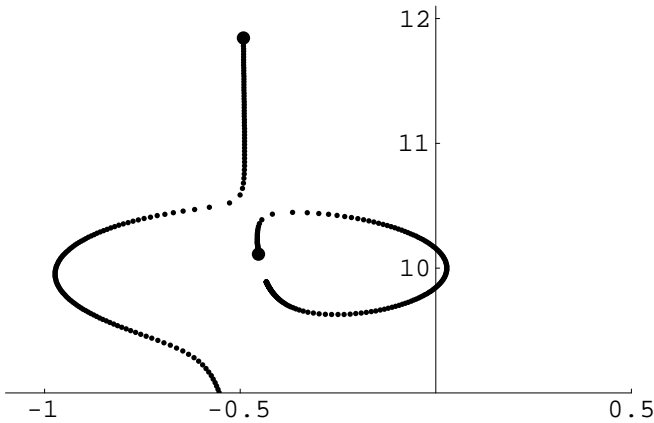

Fig. 7: Linear oscillator eigenvalues; $a = b = 1$ and intermediate damping $\delta = 0.89$.

To first show the eigenvalues without any interaction, we artificially zero the electromagnetic machine torque term by which the electrical system couples to the mechanical system. Then the torsional modes have fixed eigenvalues at the natural frequencies of the generator shaft. The movement of the subsynchronous mode as the series capacitance X_c varies and the fixed torsional modes TM1, TM2, TM3, TM4 are shown in Fig. 8. In this case, the subsynchronous and torsional eigenvalues do not interact.

When the torque term is restored, the subsynchronous and torsional modes interact in a well known way: as the subsynchronous frequency of rotor currents coincides in turn with each of the torsional mode frequencies, that torsional mode destabilizes and then restabilizes as shown in Fig. 9.

The similarity of form between each modal resonance in Fig. 9 and Fig. 3 is apparent. To further demonstrate the proximity to strong resonance in Fig. 9, we adjust the torsional mode dampings so that modes TM1, TM2, TM3 are close to their pairs of strong resonances as shown in Fig. 10. The dampings are individually tuned ($D_{EX} = 0.37$, $D_m =$

1.0, $D_{LPB} = 1.0$, $D_{LPA} = 3.0$, $D_{IP} = 2.5$, $D_{HP} = 2.0$) to obtain this special result. Although these torsional dampings are not intended to be realistic, Fig. 10 demonstrates the proximity of subsynchronous resonances in Fig. 9 to pairs of strong resonances.
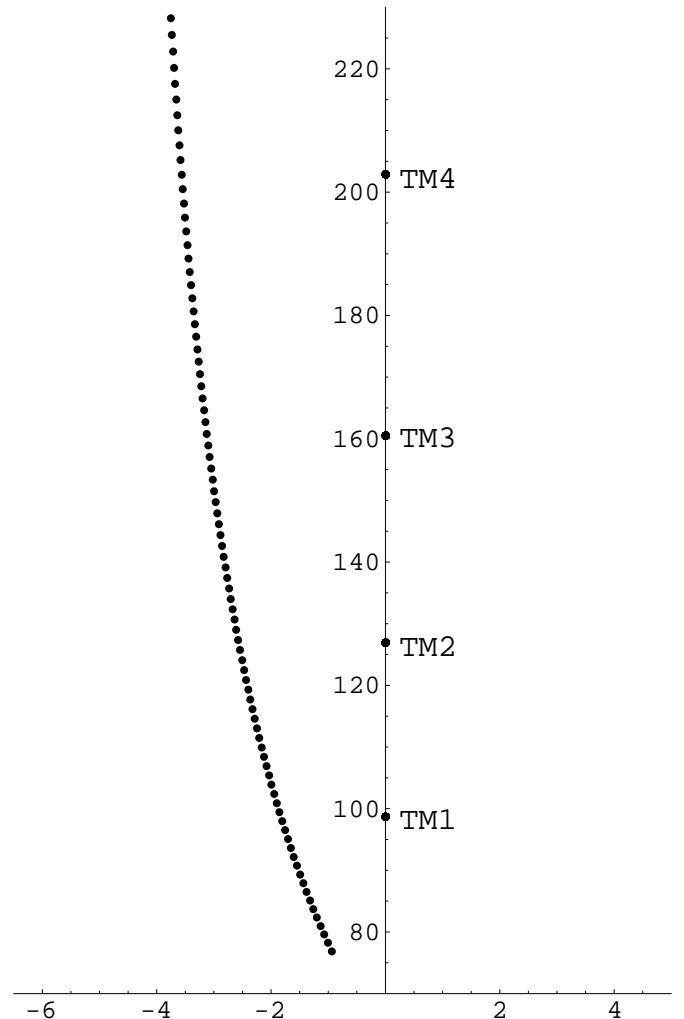

Fig. 8: Subsynchronous mode and torsional modes TM1, TM2, TM3, TM4 with no torque term and hence no interaction in the IEEE first benchmark model.

5.4 Perturbation of a complex weak resonance

Suppose that as a particular parameter t is varied a system encounters a weak resonance of complex eigenvalues. This section shows why a general perturbation of this situation leads to proximity to two strong resonances.

Assume that analytic power system differential equations have the real parameter t and the real parameter ϵ. ϵ is a real parameter controlling the perturbation. The unperturbed system has $\epsilon = 0$ and has an asymptotically stable equilibrium at $t = 0$. Let the Jacobian matrix evaluated at the stable equilibrium be $J(t, \epsilon)$. Then J is a well defined analytic function of t and ϵ near $(t, \epsilon) = (0, 0)$. We addi-
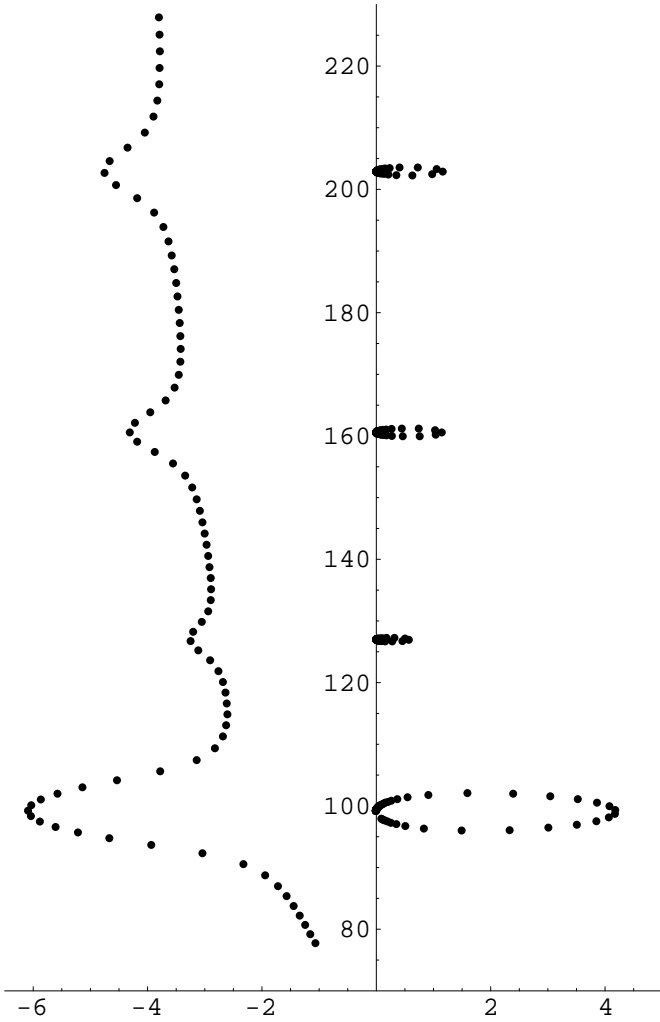
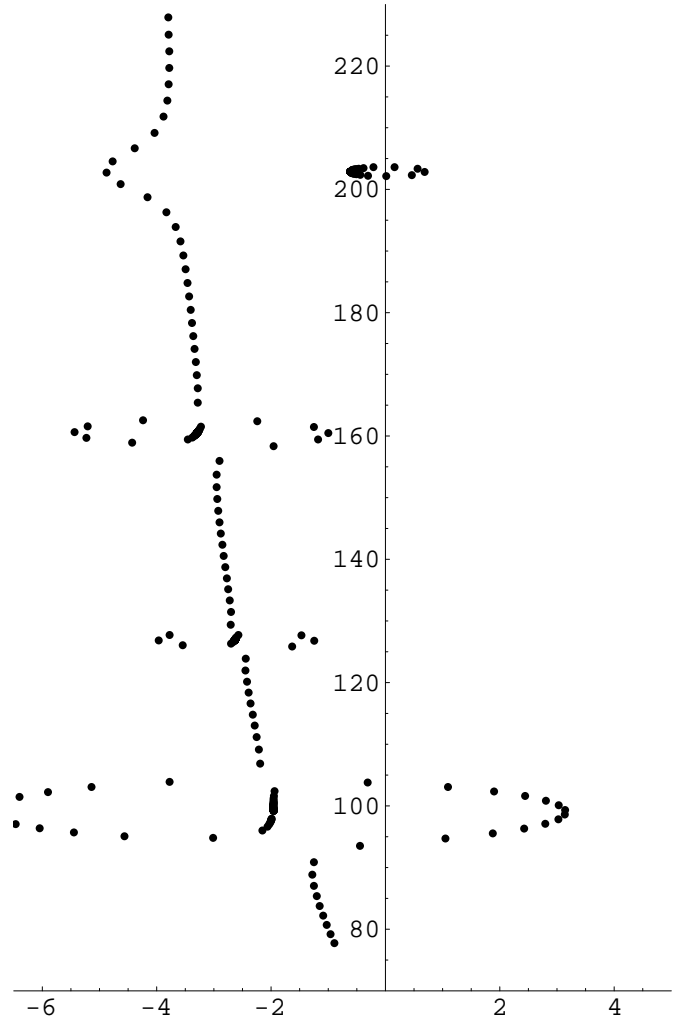Fig. 9: Subsynchronous resonances between the subsynchronous mode and torsional modes.

Fig. 10: Subsynchronous resonances perturbed with artificial torsional dampings to indicate some of the nearby pairs of strong resonances.

tionally assume that the power system differential equations depend analytically on t when t is considered to be a complex parameter. Then J is also analytic in t as a complex variable.

The unperturbed system is assumed to have a weak resonance between two complex modes at $t = 0$. In particular, exactly two eigenvalues of $J(t, 0)$ coincide at eigenvalue λ in a weak resonance[1] at $t = 0$. It follows from the analyticity of the total projection in [8, p. 74] that the two dimensional complex eigenspace corresponding to *both* of these two eigenvalues is an analytic function of t and ϵ near $(t, \epsilon) = (0, 0)$. By restricting $J(t, \epsilon)$ to this eigenspace, we obtain a 2×2 complex matrix $M(t, \epsilon)$ which captures the eigenstructure of the two modes of interest. $M(t, \epsilon)$ is an analytic function of t and ϵ near $(t, \epsilon) = (0, 0)$. We write

$$M(t, \epsilon) = \begin{pmatrix} a(t, \epsilon) & b(t, \epsilon) \\ c(t, \epsilon) & d(t, \epsilon) \end{pmatrix} \qquad (48)$$

[1] Two other complex conjugate eigenvalues of $J(t, 0)$ also coincide in a weak resonance at eigenvalue λ^* at $t = 0$.

The assumption of weak resonance at $(t, \epsilon) = (0, 0)$ implies that

$$M(0, 0) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \qquad (49)$$

Now we generalize the real parameter t to the complex parameter τ and write (with some abuse of notation) $M(\tau, \epsilon)$ for the matrix M parameterized by the complex parameter τ. This generalization to a complex parameterization with τ is convenient because complex strong resonance is codimension 1 in complex parameter space [4] and is typically encountered as τ is varied. The real parameterization by t will be restored at the end.

Define the complex function

$$\Delta(\tau, \epsilon) = (\text{Trace}\{M(\tau, \epsilon)\})^2 - 4 \, \text{Det}\{M(\tau, \epsilon)\} \quad (50)$$
$$= (a - d)^2 + 4bc \qquad (51)$$

Since the eigenvalues of M are $\frac{1}{2}(a+d) \pm \frac{1}{2}\sqrt{\Delta}$, the condition for coincident and resonant eigenvalues is $\Delta = 0$. Indeed,

the condition for strong resonance of M is $\Delta = 0$ and $bc \neq 0$ and the condition for weak resonance of M is $\Delta = 0$ and $bc = 0$. Definition (50) shows that Δ is coordinate independent. It was ensured above that J and the two dimensional complex eigenspace were analytic in τ and it follows that M and Δ are analytic in τ.

It follows from (49) that

$$\Delta(0,0) = 0 \qquad (52)$$

We want to find out how this root corresponding to a weak resonance changes under perturbation. We compute some derivatives of Δ and evaluate them at $(\tau, \epsilon) = (0,0)$. The derivative of Δ with respect to τ evaluated at $(0,0)$ is written as $\Delta_\tau|_0$. The derivatives with respect to τ are complex derivatives.

$$\Delta_\tau = 2(a-d)(a_\tau - d_\tau) + 4b_\tau c + 4bc_\tau$$
$$\Delta_\epsilon = 2(a-d)(a_\epsilon - d_\epsilon) + 4b_\epsilon c + 4bc_\epsilon$$
$$\Delta_\tau|_0 = 0$$
$$\Delta_\epsilon|_0 = 0$$
$$2A = \Delta_{\tau\tau}|_0 = 2\left[(a_\tau - d_\tau)^2 + 4b_\tau c_\tau\right]|_0$$
$$2B = \Delta_{\epsilon\epsilon}|_0 = 2\left[(a_\epsilon - d_\epsilon)^2 + 4b_\epsilon c_\epsilon\right]|_0$$
$$2H = \Delta_{\tau\epsilon}|_0 = 2\left[(a_\tau - d_\tau)(a_\epsilon - d_\epsilon) + 2b_\tau c_\epsilon + 2b_\epsilon c_\tau\right]|_0$$

Then for small (τ, ϵ) we have the Taylor series

$$\Delta(\tau, \epsilon) = A\tau^2 + 2H\tau\epsilon + B\epsilon^2 + h.o.t. \qquad (53)$$

We make the generic assumptions that $A \neq 0$ and $B \neq 0$. Then, factoring the quadratic terms

$$\Delta(\tau, \epsilon) = A^{-1}\left(A\tau + \left(H + \sqrt{H^2 - AB}\right)\epsilon\right)$$
$$\left(A\tau + \left(H - \sqrt{H^2 - AB}\right)\epsilon\right)$$
$$+ h.o.t. \qquad (54)$$

shows that, to leading order, the complex roots τ of $\Delta(\tau, \epsilon) = 0$ lie on two straight lines passing through the origin as ϵ is varied near zero. This implies that the double root at $(\tau, \epsilon) = (0,0)$ corresponding to a weak resonance typically perturbs to two strong resonances in the complex plane near zero as ϵ changes from zero. The strong resonances occur at complex values of τ which vary approximately linearly with respect to ϵ for small ϵ.

Now we restrict the parameterization from the complex parameter τ to the real parameter t. Since the two strong resonances generally lie in the complex plane off the real axis, $M(t, \epsilon)$ generally does not encounter either of the strong resonances as t varies near zero. However, for small ϵ, the proximity of the strong resonances to the origin of the complex plane implies that $M(t, \epsilon)$ does pass close to the strong resonances as t varies near zero. Thus, the power system passes close to two strong resonances at a general, small perturbation of a weak resonance.

6 Conclusions

Normal form analysis methods near strong resonance. We give examples to show that the normal form analysis coefficients C^i_{jk}, h^i_{jk} and $h2^i_{jk}z_{j0}z_{k0}$ can become very large near strong resonance between eigenvalues. Indices based on these coefficients also become large. This can occur even when the system in the original coordinates has small nonlinearity. The transformation to modal coordinates used when defining the coefficients introduces a large nonlinearity into the system and this large nonlinearity is quantified by the coefficients. In effect, the coefficients, which were intended to detect nonlinear modal interactions, also detect the strong modal resonance which occurs in the system linearization. Care is warranted in interpreting large values of these coefficients as evidence of system nonlinearity when eigenvalues are close together.

Variance of normal form analysis coefficients and indices under changes of coordinates and units. We examine how the coefficients C^i_{jk}, h^i_{jk} and $h2^i_{jk}z_{j0}z_{k0}$ vary when the power system differential equations are expressed in different coordinates or different units. Under a general coordinate change of the power system differential equations, these coefficients are not constants and do not transform as tensors. Thus the coefficients are not intrinsic to the system. This can be problematic for quantifying nonlinear interactions with these coefficients. The behavior of the coefficients under coordinate change is related to the nonuniqueness of modal coordinates and the eigenvector scaling induced by the coordinates assumed for the power system differential equations. The dependence of the indices $I1$ and $I2$ on the interaction coefficient ranking and some additional dependence of $I1$ on the coordinate system can cause these indices to vary in a nonstandard way. We suggest a new interaction coefficient (41) and index (42) which are invariant under coordinate change and intrinsic to the system. Future work should evaluate the meaning and use of these new measures.

Subsynchronous resonance and perturbations of a weak resonance. We analyze a system passing through a complex weak resonance as a single parameter is varied and show that a general perturbation leads to passing close to two strong resonances. This behavior is confirmed in a linear system of two coupled oscillators (also see [2]). We suggest that subsynchronous resonance can be understood as proximity to pair of strong resonances perturbed from a complex weak resonance and this behavior is shown by results from the IEEE first benchmark model for subsynchronous resonance. This case study on subsynchronous resonance is a start to clarifying the general implications of proximity to complex weak resonance.

Acknowledgements

Appendix

Since the arguments for the real and complex resonance cases are very similar, we only present the real resonance case. The first task is to show that for a fixed $X_0 \neq 0$, $Z_0 \nrightarrow 0$ as $\mu \to 0$. Equation (8) can be written as

$$Z_0 = U^{-1}X_0 - \begin{pmatrix} Z_0^t h2^1 Z_0 \\ Z_0^t h2^2 Z_0 \end{pmatrix} \tag{55}$$

Multiplying by U gives

$$UZ_0 = X_0 - \begin{pmatrix} Z_0^t (Uh2)^1 Z_0 \\ Z_0^t (Uh2)^2 Z_0 \end{pmatrix} \tag{56}$$

where

$$(Uh2)^1 = \frac{\epsilon}{4(1+\mu)} \begin{pmatrix} \frac{2}{(-1+\sqrt{\mu})(-1+3\sqrt{\mu})} & \frac{2}{(-1-\sqrt{\mu})(-1+\sqrt{\mu})} \\ \frac{2}{(-1-\sqrt{\mu})(-1+\sqrt{\mu})} & \frac{2}{(-1-3\sqrt{\mu})(-1-\sqrt{\mu})} \end{pmatrix}$$

$$(Uh2)^2 = \frac{\epsilon}{4(1+\mu)} \begin{pmatrix} \frac{-2+4\sqrt{\mu}}{(-1+\sqrt{\mu})(-1+3\sqrt{\mu})} & \frac{-2}{(-1-\sqrt{\mu})(-1+\sqrt{\mu})} \\ \frac{-2}{(-1-\sqrt{\mu})(-1+\sqrt{\mu})} & \frac{-2-4\sqrt{\mu}}{(-1-3\sqrt{\mu})(-1-\sqrt{\mu})} \end{pmatrix}$$

Notice that $Uh2$ is bounded as $\mu \to 0$ and that (13) shows that U is bounded as $\mu \to 0$. It follows that if $Z_0 \to 0$ as $\mu \to 0$, then letting $\mu \to 0$ in (56) implies that $X_0 = 0$, which is a contradiction. Therefore $Z_0 \nrightarrow 0$ as $\mu \to 0$.

$Z_0 \nrightarrow 0$ as $\mu \to 0$ implies that for $j = 1$ or $j = 2$, $z_{0j} \nrightarrow 0$ as $\mu \to 0$. Also (16) implies that $h2_{jj}^1 \to \infty$ as $\mu \to 0$. It is now a straightforward exercise in analysis to prove that there is a sequence μ_1, μ_2, μ_3, ... tending to zero such that $h2_{jj}^1 z_{j0} z_{j0}$ evaluated at μ_1, μ_2, μ_3, ... tends to infinity. (This conclusion is weaker than $h2_{jj}^1 z_{j0} z_{j0} \to \infty$ as $\mu \to 0$, but it is adequate for our purpose.)

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