

DAMPING AND RESONANCE IN A HIGH POWER SWITCHING CIRCUIT

Ian Dobson

Sasan Jalali

Rajesh Rajaraman

Electrical and Computer Engineering Department
University of Wisconsin, Madison, WI 53706
dobson@engr.wisc.edu

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Abstract: Switching circuits with thyristor controlled reactors are used in high power systems for static VAR control and flexible AC transmission. These circuits can exhibit highly nonlinear behavior because the thyristor switch off time depends on the circuit state. This paper shows how to understand and predict damping and resonance in a basic thyristor controlled reactor circuit used for static VAR control. The circuit damping occurs even in the absence of circuit resistance. An analytic formula for the circuit Jacobian is studied to predict resonance effects and a simple test for detecting resonance involving circuit frequencies spanning an integer harmonic is proven to be correct.

1. Introduction

As static switching circuits in power systems proliferate there is an increasing need to analyze and understand these circuits. Switching circuits involving thyristor controlled reactors such as static VAR controllers and series compensators for flexible AC transmission can exhibit damping and resonances which affect the circuit design and operation. However, these effects are poorly understood. The damping effects have been observed in simulation [2] but are often neglected because the usual linear models such as average inductor models have only resistive damping. The resonance phenomena are known to occur but are not properly accounted for in the usual models. Recent work has computed the resonances using harmonic admittance matrix techniques [1,6] and has started to develop state space methods to analyze these switching circuits [10,4,13, 7,8,3,12]. In particular, the stability of the circuits can be analytically computed using formulas for the Jacobian of the Poincaré map.

This paper shows how to use this Jacobian formula to compute and understand damping and resonance phenomena in a thyristor controlled reactor circuit which is used in reactive power control of power systems. We choose this circuit for our study because it is one of the simplest useful circuits in which these ideas can be introduced and developed. The prospects for extending these ideas to other high power switching circuits such as advanced static compensators for flexible AC transmission are good [9].

The thyristor controlled reactor circuit can be analyzed as a periodic succession of linear systems of varying dimension. The switching time between linear systems corresponding to thyristors switching on are fixed by the firing delay ϕ , but the switching times between the linear systems corresponding to thyristors switching off depend on the state (thyristors switch off when their current decreases through zero). There are both complicated and simple aspects of these circuits: This type of circuit nonlinearity leads to novel nonlinear effects such as switching time bifurcations [7,8] and Poincaré map discontinuities [12]. On the other hand, the simple structure gives opportunities for a successful exact analysis.

Sections 2 and 3 explain the basic circuit, its modeling and its classical operation. Section 4 states the analytic formula for the Jacobian of the Poincaré map of the circuit, computes the eigenvalues as a firing delay parameter is varied and discusses the nonresistive damping inherent in the circuit. Section 5 introduces a model for the Jacobian so that the Jacobian can be better understood. The Jacobian is proved to be stable for all firing delays. (Caution: stability of the Jacobian does not preclude instability via switching time bifurcations.) Section 6 explains how to compute all the halfwave symmetric orbits of the circuit and relates the resonant blowup of these orbits to eigenvalues of the Jacobian of a half cycle map being -1 . Section 7

predicts resonance when a small ambient second harmonic or third subharmonic is added to the AC voltage source and relates this to Jacobian eigenvalues on the unit circle. Section 8 gives a geometric view of the model for the Jacobian involving circular orbits on a sphere. There is a simple engineering rule of thumb involving circuit frequencies spanning an integer harmonic which is used to predict resonance in the circuit. Section 9 gives analytic conditions for circuit resonance and proves that the rule of thumb is correct.

2. System Description and Modeling

Figure 1 shows a single phase static VAR system consisting of a thyristor controlled reactor and a parallel capacitor $C = 1.5$ mF. This system is connected to an infinite bus behind a power system impedance of inductance $L_s = 0.195$ mH and resistance $R_s = 0.9$ m Ω . The controlled reactor is modeled as a series combination of an inductor $L_r = 1.66$ mH and $R_r = 31.3$ m Ω . The source voltage $u(t) = \sin \omega t$ has frequency $\omega = 2\pi 60$ rad/s and period $T = 2\pi/\omega$.

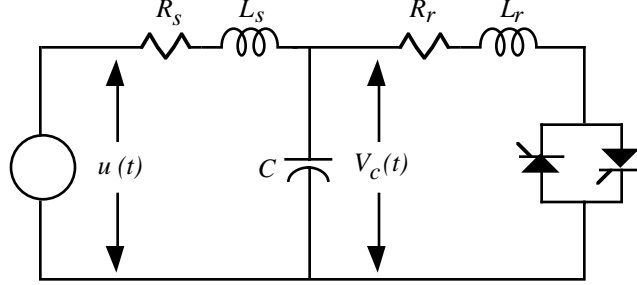


Figure 1. Single phase static VAR system.

The switching element of the thyristor controlled reactor consists of two oppositely poled thyristors which conduct on alternate half cycles of the supply frequency. A thyristor conducts current only in the forward direction, can block voltage in both directions, turns on when a firing signal is provided and turns off after a current zero. The thyristors are assumed ideal so that nonlinearities in the turn on/off of the thyristor are neglected. Indeed, the nonlinearity of the circuit only arises from the dependence of the switch off times of the thyristors on the system state. The thyristor firing pulses are supplied periodically and the system is controlled by varying the firing delay ϕ of the firing pulses. The system is studied with ϕ as an open loop control parameter. In practice a closed loop control would modify ϕ .

When a thyristor is on, the system state vector $x(t)$ specifies the thyristor controlled reactor current, capacitor voltage and the source current:

$$x(t) = \begin{pmatrix} I_r(t) \\ V_c(t) \\ I_s(t) \end{pmatrix}$$

During the conducting time of each of the thyristors, the system dynamics are described by the ON linear system:

$$\dot{x} = Ax + Bu \quad (2.1)$$

$$\text{where } A = \begin{pmatrix} -R_r L_r^{-1} & L_r^{-1} & 0 \\ -C^{-1} & 0 & C^{-1} \\ 0 & -L_s^{-1} & -R_s L_s^{-1} \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ 0 \\ L_s^{-1} \end{pmatrix}$$

During the off time of each thyristor, the circuit state is constrained to lie in the plane Z of zero thyristor current specified by $I_r = 0$. In this mode, the system state vector $y(t)$ specifies the capacitor voltage and the source current:

$$y(t) = \begin{pmatrix} V_c(t) \\ I_s(t) \end{pmatrix}$$

and the system dynamics are given by the OFF linear system

$$\dot{y} = PAQy + PBu \quad (2.2)$$

where P is the projection matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $Q = P^t$ [6,8,3].

Figure 2 describes the system dynamics as the system state evolves over a period T . A thyristor starts conducting at time ϕ_0 . This mode as described by (2.1) ends when the thyristor current goes through zero at time $\phi_0 + \sigma_1$. The non-conducting mode as described by (2.2) follows the conducting mode and continues until the next firing pulse is applied at time ϕ_1 . This starts a similar on-off cycle which lasts until the next period starts at time $\phi_0 + T$.

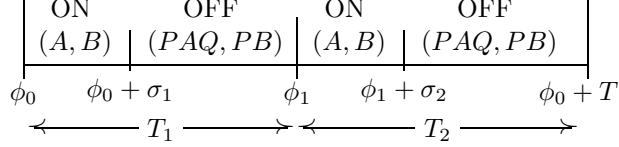


Figure 2. System dynamics from time ϕ_0 to $\phi_0 + T$.

The state at the switch on time ϕ_0 is denoted either by the vector $y(\phi_0)$ or by the vector $x(\phi_0)$. These representations of the state at the switch on time are related by

$$x(\phi_0) = Qy(\phi_0) \quad (2.3)$$

Equation (2.3) expresses the fact that the x representation of the state at a switch on is computed from the y representation by adding a new first component which has value zero.

The state at the switch off time $\phi_0 + \sigma_1$ is similarly denoted either by $x(\phi_0 + \sigma_1)$ or $y(\phi_0 + \sigma_1)$ and these are related by

$$y(\phi_0 + \sigma_1) = Px(\phi_0 + \sigma_1) \quad (2.4)$$

The matrix P in (2.4) may be thought of as projecting the vector x onto Z , where Z denotes the plane in state space of zero thyristor current.

3. Classical analysis

The classical, idealized operation of a thyristor controlled reactor is explained in Figure 3. In this figure, the gray line represents the thyristor controlled reactor voltage $V_c(t)$ (c.f. Figure 1) and the solid line represents the thyristor current.

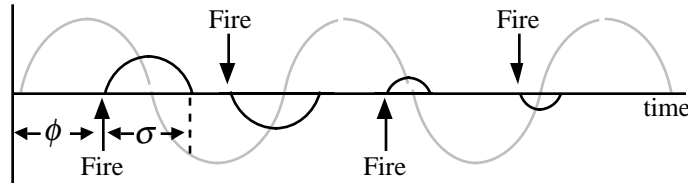


Figure 3. Classical operation of a thyristor controlled reactor.

If the thyristors are fired at the point where the voltage $V_c(t)$ is at a peak, full conduction results. The circuit then operates as if the thyristors were shorted out, resulting in a current which lags the voltage by 90 degrees. If the firing is delayed from the peak voltage, the current becomes discontinuous with a reduced fundamental component of reactive current and a reduced thyristor conduction time σ . As the firing delay angle ϕ ranges between 90° and 180° the thyristor conduction time σ ranges between 180° and 0° . The classical analysis assumes that the voltage $V_c(t)$ is a pure sinusoid.

The standard approach to modeling is to replace the thyristor controlled reactor by an average inductor model and then apply linear analysis techniques to the resulting circuit [5,11]. The average inductor model for a given firing delay ϕ is obtained by assuming a sinusoidal voltage across the thyristor controlled reactor and computing the fundamental component of the thyristor current. Then the value of average inductor for that firing delay ϕ is that which gives the same fundamental component of thyristor current. While this average inductor model approximation is usually effective for predicting steady state behavior, it fails

to capture much of the circuit nonlinearity and it breaks down when large harmonic distortions occur. Operating conditions with large harmonic distortions are documented in [1,6,15].

4. Damping and the Poincaré map Jacobian

The Poincaré map F advances the state by one period T . In particular we choose F to advance the state $y(\phi_0)$ at turn on to $F(y) = y(\phi_0 + T)$. F can be computed by integrating the system equations (2.1) and (2.2) and taking into account the coordinate changes (2.2) and (2.4) when the switchings occur [7,3]. If the circuit has a steady state periodic orbit passing through y_0 at time ϕ_0 , then y_0 is a fixed point of F :

$$F(y_0) = y_0$$

The stability of the periodic orbit is the same as the stability of y_0 and is given (except in marginal cases) by the eigenvalues of DF , the Jacobian of the Poincaré map evaluated at y_0 . The Poincaré map is differentiable except at switching time bifurcations where it is discontinuous [3,12]. Therefore we assume that the system is not exactly at a switching time bifurcation when computing the Jacobian.

According to [7,8,3], the Jacobian of the Poincaré map is

$$DF = e^{PAQ(T_2 - \sigma_2)} P e^{A\sigma_2} Q e^{PAQ(T_1 - \sigma_1)} P e^{A\sigma_1} Q \quad (4.1)$$

One of the interesting and useful consequences of formula (4.1) is that DF and the stability of the periodic orbit only depend on the state and the input via σ_1 and σ_2 . It is remarkable that (4.1) is also the formula that would be obtained for fixed switch off times σ_1 and σ_2 ; the varying switch off times introduce no additional complexity in the formula, but the nonlinearity of the circuit is clear since σ_1 and σ_2 vary as a function of $y(\phi_0)$.

If the periodic orbit is assumed to be half wave symmetric, then the two conduction lengths $\sigma_1 = \sigma_2 = \sigma$ are equal and (4.1) simplifies to

$$DF = \left(e^{PAQ(T/2 - \sigma)} P e^{A\sigma} Q \right)^2 \quad (4.2)$$

It is straightforward to compute the eigenvalue locus for halfwave symmetric orbits as the conduction time σ varies over its range of 0 to 180 degrees and the results are shown in Figure 4(a).

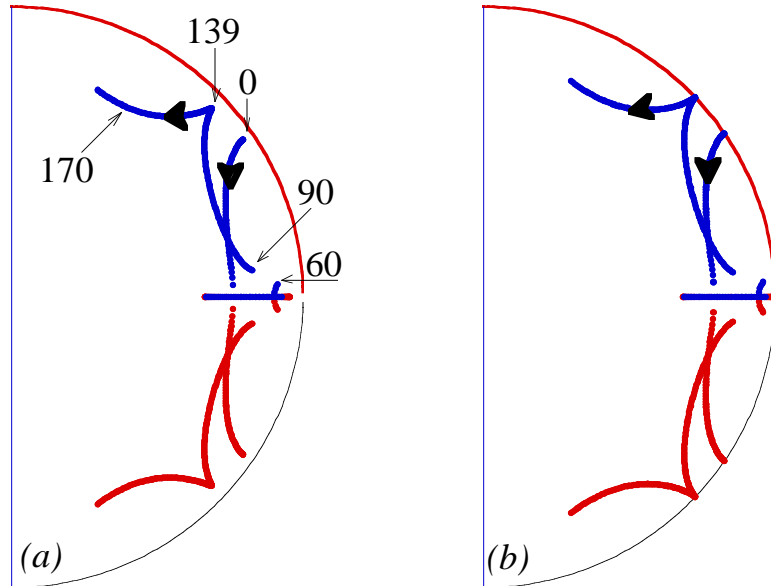


Figure 4. Eigenvalues of DH: (a) Circuit with resistance, (b) Circuit with no resistance.

When all the eigenvalues are inside the unit circle, the circuit periodic orbit is asymptotically stable and the system damps out any small perturbations. This damping cannot be entirely attributed to the

resistance in the circuit. Indeed, if the circuit resistances are set to zero then the eigenvalue locus of Figure 4(b) is obtained and the damping for most values of σ is evident. This circuit damping occurs despite the circuit impedances and thyristors being lossless; the incremental energy in a perturbation is dissipated in the source. This nonresistive damping is not accounted for in average inductor models of the circuit.

5. Model of the Jacobian

Some insight into the properties of the Jacobian formula can be obtained by considering another dynamical system whose Poincaré map is the same Jacobian. That is, we construct a useful but fictitious system that is a model for the Jacobian. For simplicity, assume half wave symmetry and consider the map H which advances the state by half of a period. Then Section 4 gives

$$DH = e^{PAQ(T/2-\sigma)}Pe^{A\sigma}Q \quad (5.1)$$

DH is also the half period map of the dynamical system given by the circuit of Figure 5, where the switch is closed at time ϕ_0 and opened at time $\phi_0 + \sigma$. Note that the circuit of Figure 5 is the thyristor controlled reactor circuit with the source removed and the thyristor replaced by an ideal switch. The states $x_\Delta(t) = (I_{r\Delta}(t), V_{c\Delta}(t), I_{s\Delta}(t))^t$ (switch on) and $y_\Delta(t) = (V_{c\Delta}(t), I_{s\Delta}(t))^t$ (switch off) of the circuit evolve according to the following equations:

$$x_\Delta(\phi_0) = Qy_\Delta(\phi_0) \quad (\text{switch closing}) \quad (5.2.1)$$

$$\dot{x}_\Delta = Ax_\Delta \quad (\text{switch ON}) \quad (5.2.2)$$

$$y_\Delta(\phi_0 + \sigma) = Px_\Delta(\phi_0 + \sigma) \quad (\text{switch opening}) \quad (5.2.3)$$

$$\dot{y}_\Delta = PAQy_\Delta \quad (\text{switch OFF}) \quad (5.2.4)$$

and it is clear from these equations that their half cycle map is DH . In general the switch current is nonzero when the switch opens at $\phi_0 + \sigma$ and the switch opening is assumed to immediately zero the inductor current. This rather nonphysical event is described in (5.2.3) by the projection of the current $x_\Delta(\phi_0 + \sigma)$ onto the plane Z by P .

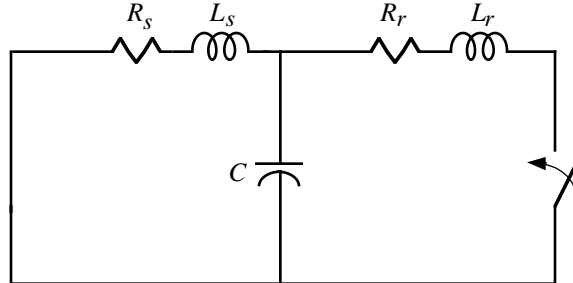


Figure 5. Circuit model for the Jacobian.

We consider the case of zero resistances so that equations (5.2.2) and (5.2.4) are simply lossless oscillators. At the beginning of the cycle the switch turns on and the on oscillation proceeds for time σ . Then the state is projected onto Z and the off oscillation proceeds for time $T/2 - \sigma$. Since we have assumed zero resistance, the oscillators are lossless and the damping in DH is entirely accounted for by the projection onto Z .

It is straightforward to use an energy or Lyapunov method to show that DH is never unstable and that its eigenvalues always lie inside or on the unit circle. Consider the incremental energy

$$E_\Delta(t) = \frac{1}{2}L_r I_{r\Delta}^2 + \frac{1}{2}L_s I_{s\Delta}^2 + \frac{1}{2}CV_{c\Delta}^2 \quad (5.3)$$

Incremental energy E_Δ is preserved at the switch closing (5.2.1) because the reactor current $I_{r\Delta}$ is zero when the switch closes (The first component of $Qy_\Delta(\phi_0)$ is always zero). At the switch opening (5.2.4), the incremental energy E_Δ decreases by the nonnegative quantity $\frac{1}{2}L_r (I_{r\Delta}(\phi_0 + \sigma))^2$ because the effect of the projection P is to zero the reactor current $I_{r\Delta}$.

In the case of zero circuit resistances, equations (5.2.2) and (5.2.4) are simply lossless oscillators. Then E_Δ is preserved at switch on, is constant while the lossless oscillators act and decreases or is preserved at switch off. Since E_Δ is a Lyapunov function for the discrete time system $y_\Delta^{k+1} = DHy_\Delta^k$, $k = 0, 1, 2, 3, \dots$, DH must be stable. If the circuit resistances are included, then E_Δ is strictly decreasing when the oscillators act and E_Δ is a strict Lyapunov function and DH is asymptotically stable.

6. Resonance

It is known that the thyristor controlled reactor circuit can exhibit a resonance for certain parameter values [1,6,7]. In particular, resonance is expected when the resonant frequency ω_{off} of the circuit when both thyristors are off and the resonant frequency ω_{on} when a thyristor is on span a harmonic of the 60 Hz fundamental frequency. Our parameter values have $\omega_{\text{off}} = 4.9 \times 60$ Hz and $\omega_{\text{on}} = 5.18 \times 60$ Hz so that the fifth harmonic is spanned. Figure 6 shows the amplitude of the capacitor voltage V_c of halfwave symmetric periodic orbits as σ is varied; the resonance occurs at $\sigma = 50.1^\circ$. This section explains and predicts the resonance by associating it with an eigenvalue of DH being -1 . Figure 6 has a region of more complex dynamics bounded by edges at which stability of the half wave symmetric solutions is lost in switching time bifurcations. These switching time bifurcations are explained in [7,8].

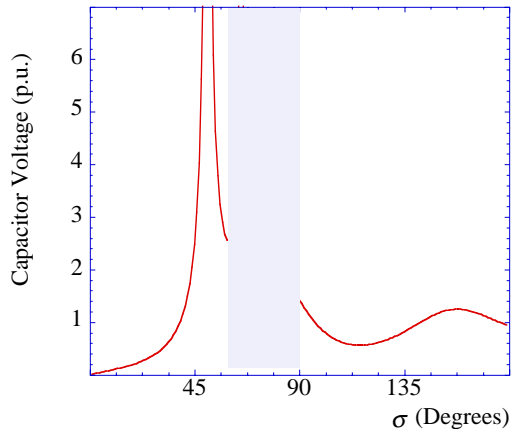


Figure 6. Capacitor voltage V_c versus firing angle σ .

When a periodic orbit is halfwave symmetric, the system state at the half cycle is equal in magnitude and opposite in sign to the system state at the beginning of the cycle. If we write $y_0 = y(\phi_0)$ for the system state at the beginning of the cycle, then this may be written in terms of the halfwave map H as

$$H(y_0) = -y_0 \quad (6.1)$$

The conduction time σ of the conducting thyristor is given by the constraint equation:

$$0 = (I - QP)x(\phi_0 + \sigma) \quad (6.2)$$

y_0 , ϕ_0 and σ corresponding to a half wave symmetric periodic orbit can be computed by solving (6.1) and (6.2) simultaneously. (Conceptually one fixes ϕ_0 and solves (6.1) and (6.2) for y_0 and σ .) Indeed, *all* halfwave symmetric solutions of the circuit can be computed by solving (6.1) and (6.2) for ϕ_0 and y_0 as σ varies from 0° to 180° . However, some of these computed solutions can be nonphysical because although the computed switch off time $\phi_0 + \sigma$ satisfies (6.2), it is not necessarily the switch off time because it need not be the *first* root of the thyristor current after ϕ_0 .

Now suppose that values of ϕ_0 and σ consistent with (6.1) and (6.2) are available. We can then compute y_0 by solving (6.1) and examine the size of y_0 to try to detect when a resonance occurs. Equation (6.1) has the form

$$-y_0 = DHy_0 + g(\sigma, \phi) \quad (6.3)$$

when written in full by integrating (2.1) and (2.2) and taking the coordinate changes at the switchings into account [7,8,3]. $g(\sigma, \phi)$ is a bounded function with terms involving the integration of the input $u(t)$ over the period. Rewriting (6.3) as

$$y_0 = -(I + DH)^{-1}g(\sigma, \phi) \quad (6.4)$$

shows that y_0 becomes unbounded as an eigenvalue of DH approaches -1 . (We assume that $g(\sigma, \phi)$ has a nonzero component along the right eigenvector corresponding to the eigenvalue -1 .) When circuit resistance is neglected, an eigenvalue of DH can equal -1 (at $\sigma = 50.1^\circ$ for our parameter values) and this corresponds to an eigenvalue approaching close to -1 when the circuit resistance is included. Thus the eigenvalues of the Jacobian DH of the circuit with no resistance can be used to predict the resonance.

Note that if an eigenvalue of DH becomes -1 then an eigenvalue of the Jacobian $DF = DH^2$ becomes 1 and this may be seen on Figure 4(b). However it is not correct to argue that the eigenvalue 1 of DF implies resonance for halfwave symmetric orbits by examining the full cycle equivalent of (6.4) because the eigenvalue 1 could also arise from DH having an eigenvalue 1 , in which case (6.4) is solvable for a bounded y_0 . However, the next section shows that DH having eigenvalue 1 implies that the orbit corresponding to this solution is resonant if a small ambient second harmonic is added to the voltage source. The reference [14] is useful background here.

7. Resonances due to ambients.

This section predicts resonance when a small ambient second harmonic or third subharmonic is added to the source voltage and eigenvalues of the halfwave Jacobian DH approach $+1$ or eigenvalues of the Jacobian DF approach certain complex roots of unity.

Suppose that the circuit has steady state periodic orbit of period T when the input is $u(t)$, the switch off times are s_{off}^1 and s_{off}^2 and the state at the start of the period is y_0 . We assume that the two switch off times satisfy the transversality conditions $\frac{\partial f}{\partial t}(s_{\text{off}}^k -) < 0$, $k = 1, 2$ where f is the thyristor current. These transversality conditions imply that the periodic orbit is not at a switching time bifurcation and that the switching times and H and F are smooth functions of sufficiently small perturbations to the initial state or the input [3].

To analyze the resonant response to the second harmonic ambient, assume that DH does not have an eigenvalue exactly at ± 1 . Let G be the map advancing the circuit state by a half period $T/2$ as a function of the initial state and the input. G has the form

$$G(y_0, u(t)) = DHy_0 + h(u(t), y_0) \quad (7.1)$$

The expression for h depends on the switch off time s_{off} which is a function of y_0 and u . However, the Jacobian simplification [7,8,3] implies that the gradient of the expression for h with respect to s_{off} vanishes. Hence if $z = O(\epsilon)$ and $v(t) = O(\epsilon)$, then

$$h(u + v, y_0 + z) = \bar{h}(u + v) + O(\epsilon^2) = \bar{h}(u) + \bar{h}(v) + O(\epsilon^2)$$

where \bar{h} is the expression for h evaluated with s_{off} determined by y_0 and u . Note that $\bar{h}(u)$ is linear in u and that the Jacobian simplification implies that $h_{y_0} = 0$ and $G_{y_0} = DH$.

Defining $\bar{G}(y_0, u) = DHy_0 + \bar{h}(u)$, we can deduce the following property of G which is needed in the sequel: If $z = O(\epsilon)$ and $v(t) = O(\epsilon)$, then

$$\begin{aligned} G(y_0 + z, u_0 + v) &= DH(y_0 + z) + h(u_0 + v, y_0 + z) \\ &= DHy_0 + DHz + \bar{h}(u_0) + \bar{h}(v) + O(\epsilon^2) \\ &= \bar{G}(y_0, u_0) + \bar{G}(z, v) + O(\epsilon^2) \end{aligned} \quad (7.2)$$

Since y_0 and u correspond to a periodic orbit we have

$$G(y_0, u(t)) = -y_0 \quad (7.3)$$

The circuit is initially in the periodic steady state specified by y_0 and u . We suppose that a small ambient second harmonic is added to the input so that the input becomes $u(t) + \epsilon \cos 2\omega t$. If ϵ is sufficiently small, we will obtain a periodic orbit near y_0 . To confirm this, define

$$K(y, \epsilon) = G(G(y, u(t) + \epsilon \cos 2\omega t), u(t + T/2) + \epsilon \cos 2\omega(t + T/2)) - y$$

Then $K(y_0, 0) = 0$ and, since DH is assumed not to have an eigenvalue exactly at ± 1 , $K_y = (G_y)^2 - I = (DH)^2 - I$ is invertible. The implicit function theorem implies that for sufficiently small ϵ , there is a smooth function $\eta(\epsilon)$ such that $\eta(0) = y_0$ and $K(\eta(\epsilon), \epsilon) = 0$. We write $z = \eta(\epsilon) - y_0$ for the deviation in y caused by the ambient and compute z . That is, we want to solve $K(y_0 + z, \epsilon) = 0$ for the displacement z to determine the eigenvalue conditions under which z is large and there is a large response. We have

$$0 = K(y_0 + z, \epsilon) = G(G(y_0 + z, u(t) + \epsilon \cos 2\omega t), \\ u(t + T/2) + \epsilon \cos 2\omega(t + T/2)) - y_0 - z$$

Since $z = \eta(\epsilon) - y_0 = O(\epsilon)$, use property (7.2) twice to obtain

$$0 = \overline{G}(\overline{G}(z, \epsilon \cos 2\omega t), \epsilon \cos 2\omega(t + T/2)) - z + O(\epsilon^2)$$

Use the definition of \overline{G} and $\cos 2\omega(t + T/2) = \cos 2\omega t$ to obtain

$$0 = (DH^2 - I)z + (I + DH)\overline{h}(\epsilon \cos 2\omega t) + O(\epsilon^2)$$

Since DH does not have eigenvalue -1 , $(I + DH)$ is invertible and hence $K(y_0 + z, \epsilon) = 0$ becomes

$$z = \epsilon(I - DH)^{-1}\overline{h}(\cos 2\omega t) + (I - DH)^{-1}O(\epsilon^2) \quad (7.4)$$

Then for sufficiently small ambient (ϵ small enough), we expect a resonance when an eigenvalue of DH approaches $+1$. (We assume that $\overline{h}(\cos 2\omega t)$ has a nonzero component along the right eigenvector corresponding to the eigenvalue $+1$.)

To analyze the response to a third subharmonic ambient, assume that DF does not have an eigenvalue exactly at 1 or $e^{\pm j2\pi/3}$. It is convenient to reuse the notation above by redefining it. Redefine G as the map advancing the circuit state by one period T as a function of the initial state and the input. G now has the form

$$G(y_0, u) = DFy_0 + h(u(t), y_0) \quad (7.5)$$

Similarly to (7.2) we can deduce that if $z = O(\epsilon)$ and $v(t) = O(\epsilon)$, then

$$G(y_0 + z, u + v) = \overline{G}(y_0, u) + \overline{G}(z, v) + O(\epsilon^2) \quad (7.6)$$

The circuit is initially in the periodic steady state specified by y_0 and u and $G(y_0, u(t)) = y_0$. A small ambient third subharmonic is added to the input so that the input becomes $u(t) + \epsilon \cos \omega_s t$ where $\omega_s = \omega/3$. If ϵ is sufficiently small, we will obtain a period 3 orbit near y_0 . To confirm this, redefine

$$K(y, \epsilon) = G(G(G(y, u(t) + \epsilon \cos \omega_s t), \\ u(t) + \epsilon \cos \omega_s(t + T)), u(t) + \epsilon \cos \omega_s(t + 2T)) - y$$

Then $K(y_0, 0) = 0$ and, since DF is assumed not to have an eigenvalue exactly at 1 or $e^{\pm j2\pi/3}$, $K_y = (G_y)^3 - I = DF^3 - I$ is invertible. The implicit function theorem implies that for sufficiently small ϵ , there is a smooth function $\eta(\epsilon)$ such that $\eta(0) = y_0$ and $K(\eta(\epsilon), \epsilon) = 0$. We write $z = \eta(\epsilon) - y_0$ and want to compute z from $K(y_0 + z, \epsilon) = 0$:

$$0 = K(y_0 + z, \epsilon) = G(G(G(y_0 + z, u(t) + \epsilon \cos \omega_s t), \\ u(t) + \epsilon \cos \omega_s(t + T)), \\ u(t) + \epsilon \cos \omega_s(t + 2T)) - y_0 - z$$

Since $z = \eta(\epsilon) - y_0 = O(\epsilon)$, use property (7.6) three times to obtain

$$0 = \overline{G}(\overline{G}(\overline{G}(z, \epsilon \cos \omega_s t), \epsilon \cos \omega_s(t + T)), \epsilon \cos \omega_s(t + 2T)) - z + O(\epsilon^2)$$

Use the definition of \bar{G} to obtain

$$0 = (DF^3 - I)z + DF^2\bar{h}(\epsilon \cos \omega_s t) \\ + DF\bar{h}(\epsilon \cos \omega_s(t+T)) + \bar{h}(\epsilon \cos \omega_s(t+2T)) + O(\epsilon^2)$$

Use the linearity of \bar{h} to rewrite as

$$0 = (DF^3 - I)z + \epsilon \mathbf{Re} \left\{ [DF^2 + DF e^{j2\pi/3} + I e^{j4\pi/3}] \bar{h}(e^{j\omega_s t}) \right\} \\ + O(\epsilon^2) \\ = (DF^3 - I)z + \epsilon (I - DF^3) \mathbf{Re} \left\{ [e^{j2\pi/3} I - DF]^{-1} \bar{h}(e^{j\omega_s t}) \right\} \\ + O(\epsilon^2)$$

That is, $K(y_0 + z, \epsilon) = 0$ becomes

$$z = \epsilon \mathbf{Re} \left\{ [e^{j2\pi/3} I - DF]^{-1} \bar{h}(e^{j\omega t}) \right\} + [I - DF^3]^{-1} O(\epsilon^2) \quad (7.7)$$

Then for sufficiently small ambient (ϵ small enough), we expect a resonance when a complex pair of eigenvalues of DF approaches $e^{\pm j2\pi/3}$. (We assume that $\bar{h}(e^{j\omega t})$ has a nonzero component along the eigenspace corresponding to the eigenvalues $e^{\pm j2\pi/3}$.)

8. Geometry of the model Jacobian

This section describes the geometry of orbits of the model Jacobian to give further insight. For simplicity, we consider only the case of zero circuit resistances. In this case, section 5 described how orbits of the model Jacobian preserve incremental energy E_Δ except at switch off. While energy is preserved, the orbits remain on an ellipsoid of constant incremental energy determined by equation (5.3). It is convenient to scale coordinates so that the ellipsoid becomes a sphere and the square of the Euclidean norm becomes the incremental energy. That is, we choose coordinates

$$a = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{L_r} I_{r\Delta} \\ \sqrt{C} V_{c\Delta} \\ \sqrt{L_s} I_{s\Delta} \end{pmatrix} \quad b = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{C} V_{c\Delta} \\ \sqrt{L_s} I_{s\Delta} \end{pmatrix}$$

so that $E_\Delta(t) = |a(t)|^2$ when the switch is on and $E_\Delta(t) = |b(t)|^2$ when the switch is off. In these coordinates, A and PAQ become the skew symmetric matrices

$$A = \begin{pmatrix} 0 & (L_r C)^{-\frac{1}{2}} & 0 \\ -(L_r C)^{-\frac{1}{2}} & 0 & (L_s C)^{-\frac{1}{2}} \\ 0 & -(L_s C)^{-\frac{1}{2}} & 0 \end{pmatrix} \quad PAQ = \begin{pmatrix} 0 & (L_s C)^{-\frac{1}{2}} \\ -(L_s C)^{-\frac{1}{2}} & 0 \end{pmatrix}$$

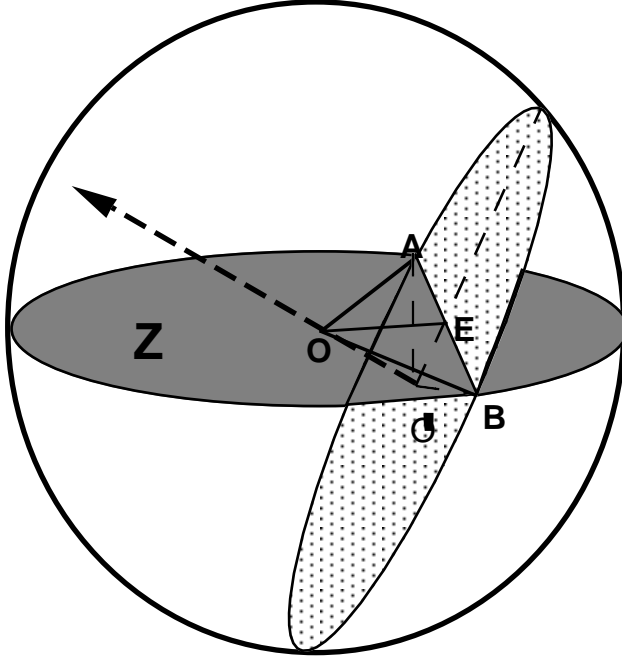


Figure 7. Dynamics of model Jacobian.

The dynamics of the model Jacobian are shown in Figure 7. The off system dynamics are confined to the plane Z of zero switch current and are a circular oscillation of frequency $\omega_{\text{off}} = \sqrt{1/L_s C}$. The on system dynamics contains a circular oscillation of frequency $\omega_{\text{on}} = \sqrt{1/(L_s C) + 1/(L_r C)}$ in which the capacitor oscillates with the parallel combination of the two inductors and a constant mode with eigenvalue zero in which the loop current through the two inductors remains constant. The oscillatory mode occurs on one of an infinite number of parallel tilted planes. The vector

$$z = \begin{pmatrix} (L_s C)^{-\frac{1}{2}} \\ 0 \\ (L_r C)^{-\frac{1}{2}} \end{pmatrix} \quad (8.1)$$

is normal to all the tilted planes and is also the eigenvector of A corresponding to the zero eigenvalue (recall that eigenvectors of skew symmetric matrices corresponding to distinct eigenvalues are orthogonal). z lies along $O'O$ in Figure 7. Let $\theta = \angle OEO'$ in Figure 7 be the angle between the tilted planes and Z . Then θ is also the angle between z and Z and

$$\cos \theta = \frac{z \cdot (1, 0, 0)}{|z|} = \frac{\omega_{\text{off}}}{\omega_{\text{on}}} \quad (8.2)$$

At the beginning of the cycle at ϕ_0 (end of the off switch mode), the state is on Z . The switch on frees the state to move in the vertical dimension ($I_{r\Delta}$ dimension). The on oscillation is a circular oscillation of frequency ω_{on} in one of the tilted planes which lasts for time σ . The tilted plane is selected by the position of the state when the switch is closed. The radius of the circular oscillation in the on system varies according to which tilted plane is selected. At switch off, the state is projected onto Z and the off oscillation proceeds for time $T/2 - \sigma$. The orbits remain on a sphere of constant incremental energy except that the projection at switch off to Z usually reduces the incremental energy. For our circuit parameters and $\sigma = 0^\circ$, an orbit makes 2.45 oscillations in the OFF system and none in the ON system whereas for $\sigma = 180^\circ$, an orbit makes 2.59 oscillations in the ON system and none in the OFF system.

9. Conditions for resonance.

Sections 7 and 8 show that the thyristor controlled reactor circuit will exhibit resonance effects if the eigenvalues of the half map DH lie near certain roots of unity. (The condition for the resonant response to a third subharmonic ambient is given in terms of $DF = DH^2$.) When component values are specified, it is practical to check for resonance by numerically computing the eigenvalues of DH as σ varies from 0 to $T/2$ and observing whether the eigenvalues approach certain roots of unity. However, it is much better to derive the analytic results below describing whether and where DH has eigenvalues on the unit circle. Surprisingly, some of the analytic results simplify even further: Consideration of the average inductor model and case studies led in [1] to a rule of thumb for the thyristor controlled reactor circuit that if ω_{off} and ω_{on} span an odd multiple of ω , resonance will occur for some value of σ . This simple and useful rule is indeed correct and a precise version of the rule is proved from the analytic criteria. The resonance response to second harmonic ambient is predicted by a similar rule.

Theorem 1 describes the conditions for the eigenvalues of DH to lie on parts of the unit circle and is proved in the Appendix. The degenerate case of the ON system having eigenvectors in the plane Z is excluded.

Theorem 1 *Consider the thyristor controlled reactor circuit with no resistance and assume that A has no eigenvectors in Z . Then*

(A) DH has an eigenvalue of -1 iff

$$(i) \quad \omega_{\text{on}}\sigma \neq 0 \pmod{2\pi} \quad \text{and} \quad \frac{\omega_{\text{on}}}{\omega_{\text{off}}} = \tan\left(\frac{\omega_{\text{off}}(T/2 - \sigma)}{2}\right) \tan\left(\frac{\omega_{\text{on}}\sigma}{2}\right)$$

$$\text{or } (ii) \quad \omega_{\text{on}}\sigma = 0 \pmod{2\pi} \quad \text{and} \quad \omega_{\text{off}}(T/2 - \sigma) = \pi \pmod{2\pi}.$$

(B) DH has an eigenvalue $+1$ iff

$$(i) \quad \omega_{\text{on}}\sigma \neq 0 \pmod{2\pi} \quad \text{and} \quad -\frac{\omega_{\text{on}}}{\omega_{\text{off}}} = \cot\left(\frac{\omega_{\text{off}}(T/2 - \sigma)}{2}\right) \tan\left(\frac{\omega_{\text{on}}\sigma}{2}\right)$$

$$\text{or } (ii) \quad \omega_{\text{on}}\sigma = 0 \pmod{2\pi} \quad \text{and} \quad \omega_{\text{off}}(T/2 - \sigma) = 0 \pmod{2\pi}.$$

(C) DH has eigenvalues at $e^{\pm j\pi/m}$, $m \geq 1$ iff

$$\omega_{\text{on}}\sigma = 0 \pmod{2\pi} \quad \text{and} \quad \omega_{\text{off}}(T/2 - \sigma) = \pi/m \pmod{2\pi}.$$

We examine which of these conditions are generically satisfied. For given circuit parameters, ω_{on} and ω_{off} are constants and σ is the only variable. Conditions $A(i)$ and $B(i)$ require one equation to be satisfied whereas conditions $A(ii)$, $B(ii)$ and C require two independent equations to be satisfied. Thus we expect that conditions $A(i)$ and $B(i)$ can be generically satisfied for some σ whereas conditions $A(ii)$, $B(ii)$ and C would only be satisfied for some σ for exceptional values of circuit parameters.

Theorem 2 further simplifies the generically occurring conditions $A(i)$ and $B(i)$ by reducing them to rules of thumb for resonance. A slightly more detailed result than Theorem 2 specifying the number of resonance conditions is stated and proved in the Appendix.

Theorem 2 *Consider the thyristor controlled reactor circuit with no resistance and assume that A has no eigenvectors in Z .*

If $\omega_{\text{on}}\sigma \neq 0 \pmod{2\pi}$, then

(A) DH has an eigenvalue at -1 iff ω_{off} and ω_{on} span an odd multiple of ω .

(B) DH has an eigenvalue at $+1$ iff ω_{off} and ω_{on} span an even multiple of ω .

10. Conclusions

This paper studies damping and resonance phenomena in a basic thyristor controlled reactor circuit for static VAR control by exploiting a simple exact formula for the Jacobian of the Poincaré map. The nonresistive damping of the circuit is described and resonance effects are associated with eigenvalues of Jacobians lying near ± 1 or complex roots of unity. If circuit resistance is neglected, analytic formulas are derived which describe the conditions for eigenvalues to be at ± 1 or complex roots of unity. Hence a simple test for detecting resonance involving circuit frequencies spanning an integer harmonic is proven to be correct. This exact analysis of resonance is unusual in a nonlinear system. We expect that much of the analysis in this paper could be generalized to more complex circuits containing thyristor controlled reactors such as advanced

series compensators used in flexible AC transmission. The ability to predict and understand damping and resonance in these circuits should be useful in the design of these devices.

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Appendix: Proofs of resonance conditions

Consider the thyristor controlled reactor circuit with all resistances zero. The Jacobian of the half cycle map is $DH = e^{PAQ(T/2-\sigma)}Pe^{A\sigma}Q$. Assume that A has no eigenvectors in Z and write $\beta = \omega_{\text{on}}\sigma$ for the angle traversed in the ON system. We derive conditions under which the half cycle map DH has eigenvalues on the unit circle.

Lemmas 1, 2 and 3 prove Theorem 1 by describing all the ways for DH to have eigenvalues on the unit circle.

Lemma 1

(a) *If $\beta \neq 0 \pmod{2\pi}$, then the ON system maps an initial vector $a(0) = (0, r \sin(\alpha/2), r \cos(\alpha/2))$ in Z to a vector $a(\sigma)$ in Z iff*

$$\tan(\beta/2) \cos \theta = \tan(\alpha/2) \quad (\text{A.1})$$

Moreover, $a(\sigma) = (0, -r \sin(\alpha/2), r \cos(\alpha/2))$ and α is the angle subtended at the origin by $a(0)$ and $a(\sigma)$ in the plane Z .

(b) *$\beta = 0 \pmod{2\pi}$ iff the ON system maps all vectors in Z into Z .*

Proof : ((a) \Rightarrow) Suppose the ON system maps $a(0) = (0, r \sin(\alpha/2), r \cos(\alpha/2))$ to $a(\sigma) = (0, a_2, a_3)$ in Z . Recall that $z = ((L_s C)^{-\frac{1}{2}}, 0, (L_r C)^{-\frac{1}{2}})$ is the eigenvector of A with zero eigenvalue. Since the component of $a(t)$ in the direction of z is constant, $a(0).z = a(\sigma).z$ and hence $a_3 = r \cos(\alpha/2)$. The ON system preserves the Euclidean norm so that $r^2 = |a(0)|^2 = |a(\sigma)|^2 = a_2^2 + a_3^2 = a_2^2 + r^2 \cos^2(\alpha/2)$ and $a_2 = \pm r \sin(\alpha/2)$. If $a_2 = r \sin(\alpha/2)$, then $a(0) = a(\sigma)$, and hence $\beta = 0 \pmod{2\pi}$ which contradicts the assumption. Therefore $a_2 = -r \sin(\alpha/2)$ and $a(\sigma) = (0, -r \sin(\alpha/2), r \cos(\alpha/2))$. In Figure 7, $a(0)$ is point A and $a(\sigma)$ is point B and $\alpha = \angle AOB$ is the angle subtended at the origin by $a(0)$ and $a(\sigma)$ in the plane Z . Moreover, $\theta = \angle OEO'$ and $\beta = \angle AO'B$. The angles $\angle O'EA$, $\angle EO'O$, $\angle OEA$ are right angles and $\beta/2 = \angle AO'E$ and $\alpha/2 = \angle AOE$. Now use the right angled triangles $O'EA$, $EO'O$ and OEA to obtain

$$\tan(\beta/2) = \frac{EA}{O'E} = \frac{EA}{OE \cos \theta} = \frac{\tan(\alpha/2)}{\cos \theta}$$

((a) \Leftarrow) It can be verified that the ON system maps $a(0) = (0, r \sin(\alpha/2), r \cos(\alpha/2))$ to

$$a(\sigma) = \begin{pmatrix} r \operatorname{cosec} \theta \sin(\alpha/2) \sin \beta + r \cot \theta \cos(\alpha/2) (\cos \beta - 1) \\ r \sin(\alpha/2) \cos \beta - r \cos \theta \cos(\alpha/2) \sin \beta \\ r \sec \theta \sin(\alpha/2) \sin \beta + r \cos(\alpha/2) \cos \beta \end{pmatrix}$$

Using (A.1) and simplifying, $a(\sigma) = (0, -r \sin(\alpha/2), r \cos(\alpha/2))$ is in Z .

((b) \Rightarrow) If $\beta = 0 \pmod{2\pi}$, then $e^{A\sigma}$ is the identity.

((b) \Leftarrow) Suppose that the ON system maps each vector in Z into Z and $\beta \neq 0 \pmod{2\pi}$. Then by choosing any α that does not satisfy (A.1), it is clear from Lemma 1(a) that $a(0) = (0, \sin(\alpha/2), \cos(\alpha/2))$ will not be mapped to Z by the ON system and a contradiction has been obtained. ■

Lemma 2 *Assume that $\beta \neq 0 \pmod{2\pi}$. Then*

(a) *DH has an eigenvalue at -1 iff*

$$\frac{\omega_{\text{on}}}{\omega_{\text{off}}} = \tan \left(\frac{\omega_{\text{off}}(T/2 - \sigma)}{2} \right) \tan \left(\frac{\omega_{\text{on}}\sigma}{2} \right), \quad (\text{A.2})$$

(b) *DH has an eigenvalue at $+1$ iff*

$$-\frac{\omega_{\text{on}}}{\omega_{\text{off}}} = \cot \left(\frac{\omega_{\text{off}}(T/2 - \sigma)}{2} \right) \tan \left(\frac{\omega_{\text{on}}\sigma}{2} \right). \quad (\text{A.3})$$

Proof : We will prove the assertion for (a); the proof for (b) is similar.

((a) \Rightarrow) Let $b(0) = (\sin(\alpha/2), \cos(\alpha/2))$ be the eigenvector of DH corresponding to the eigenvalue -1 . $|b(T/2)| = |-b(0)| = |b(0)|$ and the preservation of the norm by the OFF system imply that $|Pa(\sigma)| = |b(\sigma)| = |b(0)|$. Also $|a(0)| = |Qb(0)| = |b(0)|$ and the preservation of the norm by the ON system imply that $|a(\sigma)| = |b(0)|$. Hence $|a(\sigma)| = |Pa(\sigma)|$ and $a(\sigma) \in Z$. $a(0) = Qb(0) = (0, \sin(\alpha/2), \cos(\alpha/2))$ and Lemma 1(a) imply that $a(\sigma) = (0, -\sin(\alpha/2), \cos(\alpha/2))$. Thus $b(\sigma) = Pa(\sigma) = (-\sin(\alpha/2), \cos(\alpha/2))$. But $b(T/2) = -b(0) = (-\sin(\alpha/2), -\cos(\alpha/2))$ so that the angle swept by the OFF system is $\pi - \alpha \pmod{2\pi}$. Therefore $\omega_{\text{off}}(T/2 - \sigma) = \pi - \alpha \pmod{2\pi}$ and $\cot(\alpha/2) = \tan(\omega_{\text{off}}(T/2 - \sigma)/2)$. The result follows from (A.1) of Lemma 1(a) and $\cos\theta = \omega_{\text{off}}/\omega_{\text{on}}$ from (8.2).

((a) \Leftarrow) Suppose (A.2) is true. Since $\beta \neq 0 \pmod{2\pi}$, we can choose $\alpha \neq 0 \pmod{2\pi}$ such that (A.1) is satisfied. Let $b(0) = (\sin(\alpha/2), \cos(\alpha/2))$. Then $a(0) = Qb(0) = (0, \sin(\alpha/2), \cos(\alpha/2))$ and Lemma 1(a) implies that $a(\sigma) = (0, -\sin(\alpha/2), \cos(\alpha/2))$. Now $b(\sigma) = Pa(\sigma) = (-\sin(\alpha/2), \cos(\alpha/2))$. (A.2) and (A.1) imply that $\tan(\alpha/2)\tan(\omega_{\text{off}}(T/2 - \sigma)/2) = 1$ and hence $\omega_{\text{off}}(T/2 - \sigma) = \pi - \alpha \pmod{2\pi}$. That is, the angle swept by the OFF system is $\pi - \alpha \pmod{2\pi}$. It follows that $b(T/2) = (-\sin(\alpha/2), -\cos(\alpha/2)) = -b(0)$ so that $b(0)$ is an eigenvector of DH with eigenvalue -1 .

Lemma 3 $\beta = 0 \pmod{2\pi}$ iff both eigenvalues of the half cycle map DH are on the unit circle. In particular, the eigenvalues are located at $e^{\pm j\omega_{\text{off}}(T/2 - \sigma)}$.

Proof : (\Rightarrow) If $\beta = 0 \pmod{2\pi}$, then $e^{A\sigma}$ is the identity and $DH = e^{PAQ(T/2 - \sigma)}$ has eigenvalues at $e^{\pm j\omega_{\text{off}}(T/2 - \sigma)}$ on the unit circle.

(\Leftarrow) If DH has both eigenvalues on the unit circle, then DH preserves the norm and $|b(0)| = |b(T/2)|$ for all $b(0)$. Suppose that the ON system maps $a(0) \in Z$ to $a(\sigma) \notin Z$. Then $|Pa(\sigma)| < |a(\sigma)|$ and $|b(T/2)| = |b(\sigma)| = |Pa(\sigma)| < |a(\sigma)| = |a(0)| = |Qb(0)| = |b(0)|$ is a contradiction. Therefore the ON system maps all $a(0) \in Z$ into Z and the result follows from Lemma 1(b). ■

Theorem 2 part (A) is now proved by demonstrating that if ω_{off} and ω_{on} span k odd multiples of ω , then (A.2) has k roots as σ varies from 0 to $T/2$ and that if ω_{off} and ω_{on} span k even multiples of ω , then (A.3) has k roots as σ varies from 0 to $T/2$. The proof of part (B) of Theorem 2 demonstrates that (A.3) has as many roots as the number of even multiples of ω spanned by ω_{off} and ω_{on} and is omitted since it is very similar to the proof of part (A).

Lemma 4 Let g and h be real differentiable functions on the interval $[0, 1]$. Suppose that g and h have the same sign at critical points of g and at the end points 0, 1. Precisely, either $g(\delta)h(\delta) > 0$ or $g(\delta) = h(\delta) = 0$, when $g'(\delta) = 0$ or $\delta = 0, 1$. If all the critical points of g are isolated, the number of roots of h in $[0, 1] \geq$ number of roots of g in $[0, 1]$.

Proof : Since the critical points of g are isolated, each root of g is at a critical point of g or at 0 or at 1 or occurs between successive critical points of g or between 0 and the smallest critical point or between the largest critical point and 1 or, if there are no critical points, between 0 and 1. Write δ_1 for a critical point of g (or 0) and δ_2 for the successive critical point of g (or 1).

We assert that if there is a root of g strictly between δ_1 and δ_2 , then $g(\delta_1)g(\delta_2) < 0$. For suppose that $g(\delta_1)g(\delta_2) \geq 0$ and $g(\delta_r) = 0$ with $\delta_1 < \delta_r < \delta_2$. Consider the case of $g(\delta_1) \geq 0$ and $g(\delta_2) \geq 0$; the case of $g(\delta_1) \leq 0$ and $g(\delta_2) \leq 0$ is similar. Choose $\delta_m \in [\delta_1, \delta_2]$ such that $g(\delta_m) = \min\{g(\delta) \mid \delta \in [\delta_1, \delta_2]\}$. Then $g(\delta_m) \leq g(\delta_r) = 0$. Moreover, $\delta_m \neq \delta_1$ because $\delta_m = \delta_1$ implies $0 = g(\delta_r) \geq g(\delta_m) = g(\delta_1) \geq 0$ and $g(\delta_1) = g(\delta_m) = 0$. Then Rolle's theorem implies that there is a critical point of g in (δ_1, δ_m) which contradicts $\delta_2 \geq \delta_m$ being the next critical point after δ_1 . Similarly, $\delta_m \neq \delta_2$. But then $\delta_m \in (\delta_1, \delta_2)$ and $g'(\delta_m) = 0$ contradicts δ_2 being the next critical point after δ_1 .

Therefore if there is a root of g strictly between δ_1 and δ_2 , then $g(\delta_1)g(\delta_2) < 0$. But by hypothesis, $g(\delta_1)g(\delta_2)h(\delta_1)h(\delta_2) > 0$, and hence $h(\delta_1)h(\delta_2) < 0$ and h must also have a root in (δ_1, δ_2) . Further, if $g(\delta_1) = 0$, then by hypothesis, $h(\delta_1) = 0$ and if $g(\delta_2) = 0$ then $h(\delta_2) = 0$. Thus for each root of g there is a distinct root of h and the number of roots of h in $[0, 1] \geq$ number of roots of g in $[0, 1]$. ■

Lemma 5 If ω_{off} and ω_{on} span k odd multiples of ω , then (A.2) has k roots as σ varies from 0 to $T/2$.

Proof : First multiply both sides of (A.2) by $\pi/2$, use the following changes in notation: $\omega_{\text{off}}\pi/2 = b_1\omega$, $\omega_{\text{on}}\pi/2 = b_2\omega$, $\sigma = \delta T/2$, and rearrange to obtain $h(\delta) = 0$ where

$$h(\delta) = (b_1 + b_2) \cos((b_2 - b_1)\delta + b_1) + (b_2 - b_1) \cos((b_1 + b_2)\delta - b_1).$$

Also define

$$g(\delta) = (b_1 + b_2) \cos((b_2 - b_1)\delta + b_1)$$

Note that $b_1 > 0$, $b_2 > 0$, $b_1 \neq b_2$ and $0 \leq \delta \leq 1$.

Let δ^* be either a critical point of h or 0 or 1. If δ^* is a critical point of h , then $h'(\delta^*) = 0$ implies that $\sin((b_1 + b_2)\delta^* - b_1) = -\sin((b_2 - b_1)\delta^* + b_1)$ and hence $\cos((b_1 + b_2)\delta^* - b_1) = \pm \cos((b_2 - b_1)\delta^* + b_1)$. Thus $h(\delta^*) = g(\delta^*)(1 \pm (b_2 - b_1)/(b_1 + b_2))$. Since $b_1 + b_2 \pm (b_2 - b_1) > 0$, either $h(\delta^*)g(\delta^*) > 0$ or $h(\delta^*) = g(\delta^*) = 0$, so that g and h have the same sign at critical points of h . If δ^* is either 0 or 1, $h(\delta^*) = g(\delta^*)(1 + (b_2 - b_1)/(b_1 + b_2))$. Thus, either $h(\delta^*)g(\delta^*) > 0$ or $h(\delta^*) = g(\delta^*) = 0$. The critical points of h are isolated and Lemma 4 implies that the number of roots of g in $[0, 1] \geq$ number of roots of h in $[0, 1]$.

Let δ_* either be a critical point of g or 0 or 1. If δ_* is a critical point of g , then $g(\delta_*) = \pm(b_1 + b_2)$ and $h(\delta_*)g(\delta_*) = (b_1 + b_2)(b_1 + b_2 \pm (b_2 - b_1) \cos((b_1 + b_2)\delta_* - b_1)) > 0$. If δ_* is either 0 or 1, then we have already shown that either $h(\delta_*)g(\delta_*) > 0$ or $h(\delta_*) = g(\delta_*) = 0$. Apply Lemma 4 again to conclude that the number of roots of h in $[0, 1] \geq$ number of roots of g in $[0, 1]$.

Thus the number of roots of h in $[0, 1]$ is equal to the number of roots of g in $[0, 1]$. But the argument $(b_2 - b_1)\delta + b_1$ of the cosine term in $g(\delta)$ is monotonic with respect to δ and hence the number of roots of $g(\delta) = (b_1 + b_2) \cos((b_2 - b_1)\delta + b_1)$ in $[0, 1]$ is the number of odd multiples of $\pi/2$ spanned by b_1 and b_2 , which is the same as the number of odd multiples of ω spanned by ω_{off} and ω_{on} . ■

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