Thirty-eighth Hawaii International Conference on System Sciences, January 2005, Big Island, Hawaii. 

# Branching process models for the exponentially increasing portions of cascading failure blackouts

Ian Dobson
ECE Department
University of Wisconsin
Madison WI 53706 USA
dobson@engr.wisc.edu

Benjamin A. Carreras
Oak Ridge National Laboratory
Oak Ridge TN 37831 USA
carrerasba@ornl.gov

David E. Newman
Physics Department
University of Alaska
Fairbanks AK 99775 USA
ffden@uaf.edu

## Abstract

*We introduce branching process models in discrete and continuous time for the exponentially increasing phase of cascading blackouts. Cumulative line trips from real blackout data have portions consistent with these branching process models. Some initial calculations identifying parameters and using a branching process model to estimate blackout probabilities are illustrated.*

## 1. Introduction

We aim to capture gross features of large, cascading failure blackouts using probabilistic branching process models. Galton-Watson and Markov branching processes are related to the timing of failures and this extends previous work that models the evolution but not the timing of the blackout failures with Galton-Watson branching processes [5]. This overall approach is complementary to the traditional and useful detailed analysis of blackouts and offers a number of possibilities for understanding and monitoring the risk of large blackouts.

Section 2 examines transmission line failure data from three recent North American blackouts for exponentially increasing portions and estimates the exponents of the exponential increases. Section 3 considers Markov branching process models in discrete and continuous time that reflect the exponential increase [1, 8] and suggests methods of identifying branching process parameters. Section 4 shows sample calculations of how a branching process model could be used to explore the likelihood of a particular blackout occurring and the value of including real time data on the cumulative number of line trips in estimates of the blackout propagation.

## 2. Blackout data

This section examines cumulative high voltage line trips in observed blackout data from the July and August 1996 WSCC blackouts [9, 11] and the August 2003 Eastern interconnect blackout [10].

It is supposed that are three phases to the blackout. The effect of the first phase is summarized as an initial disturbance that causes a certain number of line trips at the beginning of the cascading phase. In the second, cascading phase, the cascading process can cause exponentially increasing cumulative line trips. In the final phase, the cascading process saturates and the blackout starts to slow down and converge to its final extent. The identification of the boundaries between the blackout phases is done by inspection of the data.

For each blackout, we plot the cumulative line trips with respect to time to examine the overall trajectory of the blackout. If there is an exponentially increasing phase, then this should appear as a straight line portion in a plot of the logarithm of the cumulative line trips with respect to time and the slope of the line gives the exponent of the exponential growth.

There is no attempt to filter the data by, for example, combining trips of parallel lines. Generator trips are not included in the data. Trips of lines of different ratings are counted in the same way. These assumptions are made for simplicity in order to make a first analysis of the data from this new perspective.

### 2.1. July 1996 WSCC blackout

Figure 1 shows cumulative line trips as function of time extracted from the 1996 NERC system disturbance report [9], page 28. The lines tripped include lines of ratings from 120 kV to 500 kV. The initial disturbance is taken as 2 line trips at 14:24 MDT. Examining the logarithm of the cumulative line trips in excess of 2 in Figure 2 suggests an exponential growth between times 14:24 to 14:31 MDT. The exponent of the exponential growth is $\mu \approx 0.47$ min$^{-1}$. This corresponds to multiplication of the cumulative line trips by a factor of 1.6 every minute.

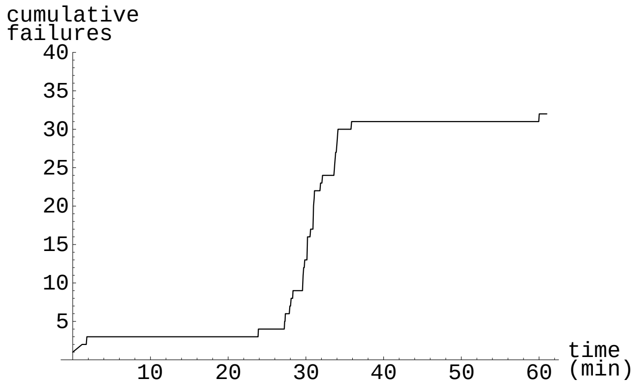


**Figure 1. Cumulative line trips in WSCC July 1996 blackout. Time scale is minutes after 14:00 MDT.**

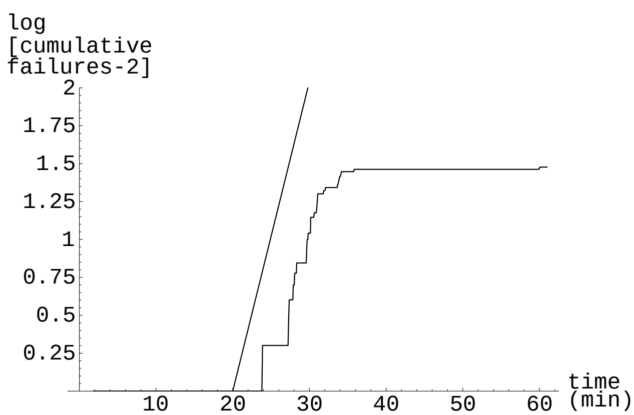


**Figure 2. Log[cumulative line trips in excess of 2] in WSCC July 1996 blackout. The straight line growth corresponds to $1.6^{\rm time}$. Time scale is minutes after 14:00 MDT.**

### 2.2. August 1996 WSCC blackout

Figure 3 shows cumulative line trips as function of time extracted from the 1996 NERC system disturbance report [9], page 38. The initial disturbance is taken as 2 line trips at 14:46 PDT. Examining the logarithm of the cumulative line trips in excess of 2 in Figure 4 suggests an exponential growth between times 13:46 to 13:49 PDT. The exponential growth is somewhat less clear cut than in the July 1996 blackout because it evolves quickly in only a few jumps. The exponent of the exponential growth is $\mu \approx 1.4$ min$^{-1}$. This corresponds to multiplication of the cumulative line trips by a factor of 4 every minute.

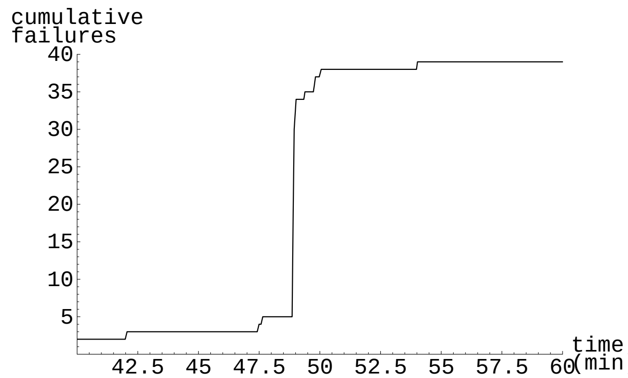


**Figure 3. Cumulative line trips in WSCC August 1996 blackout. Time scale is minutes after 15:00 PDT.**

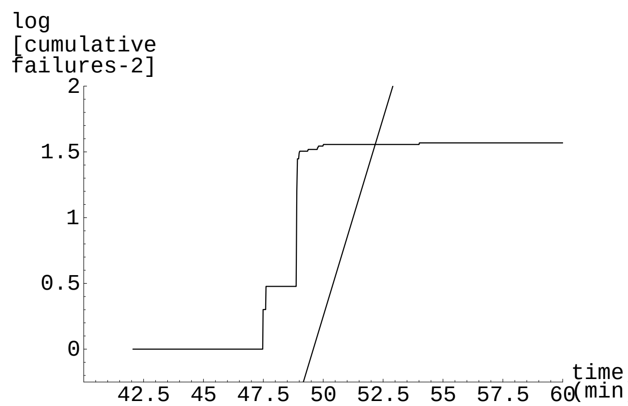


**Figure 4. Log[cumulative line trips in excess of 2] in WSCC August 1996 blackout. The straight line growth corresponds to $4^{\rm time}$. Time scale is minutes after 15:00 PDT.**

### 2.3. August 2003 Eastern interconnect blackout

Figure 5 shows cumulative line and transformer trips as function of time reprinted from the final blackout report [10]. Since the data underlying Figure 5 is not yet available to us for study, we digitized by hand the cumulative line and transformer trips curve in Figure 5 to obtain approximate data and then replotted the logarithm of the cumulative trips as Figure 6. One way to parse the data in Figure 6 is to consider a slowly cascading phase from time

5.5 to 8.5 and a fast cascading phase from time 8.5 to 9.5, and then saturation of the fast cascading phase. The slow cascading phase fits an exponential more approximately. The slow cascading phase has exponent of the exponential growth $\mu \approx 0.34$ min$^{-1}$. This corresponds to multiplication of the cumulative line trips by a factor of 1.4 every minute. The fast cascading phase has exponent of the exponential growth $\mu \approx 2.9$ min$^{-1}$. This corresponds to multiplication of the cumulative line trips by a factor of 18 every minute.

There are other ways of parsing the data in Figure 6; one could simply fit the data with a single exponential cascading phase from time 5.5 to 9.5. One reason for preferring the fit with two cascading phases considered in the preceding paragraph in an initial exploration of the data is that power system experts identified two cascading phases [10]. However, Figure 6 raises the question of whether the data is best fit by one or two cascading phases.

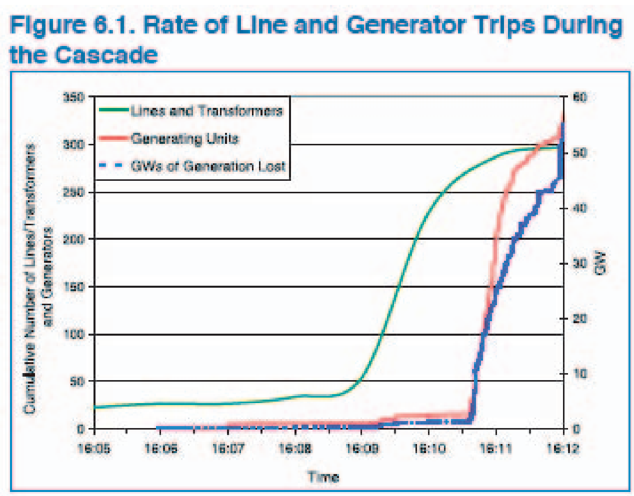


**Figure 5. Cumulative line and transformer trips in August 2003 blackout. Reprinted from [10].**

We conclude that several recent North American blackouts show a region or regions of exponential increase in cumulative line failures.

## 3. Branching process models

Branching process models are an obvious choice of stochastic model to capture the gross features of cascading blackouts because they have been developed and applied to other cascading processes such as genealogy, epidemics and cosmic rays [8]. The first suggestion to apply branching processes to blackouts appears to be in [5].

There are more specific arguments justifying branching processes as good approximations to some of the gross features of cascading blackouts. An idealized probabilistic model of cascading failure [7, 4] describes with analytic formulas the statistics of a cascading process in which component failures weaken and further load the system so that subsequent failures are more likely. It is known that this cascade model and variants of it can be well approximated by a Galton-Watson branching process with each failure giving rise to a Poisson distribution of failures in the next stage. [5, 6]. Moreover, some features of this cascade model are consistent with results from cascading failure simulations [2, 4]. All of these models can show power law regions in the distribution of failure sizes or blackout sizes consistent with NERC data [3].

All the cascading failure models and branching processes considered above make no reference to the time of failures; the failures are produced in successive stages without reference to the time of each stage. This raises the issues of how to relate the stages to data that arises in real time and whether a branching process model in continuous time can be applied. We consider three possible approaches below. The first two approaches consider a Galton-Watson branching process in which the failures occur in stages and the third approach considers a continuous time branching process. All the standard facts quoted below about branching processes are available in [1, 8].

### 3.1. Galton-Watson branching process with variable time between stages

The Galton-Watson branching process is assumed to have each failure generate failures in the subsequent stage according to a distribution with mean $\lambda$. $\lambda$ is a measure of

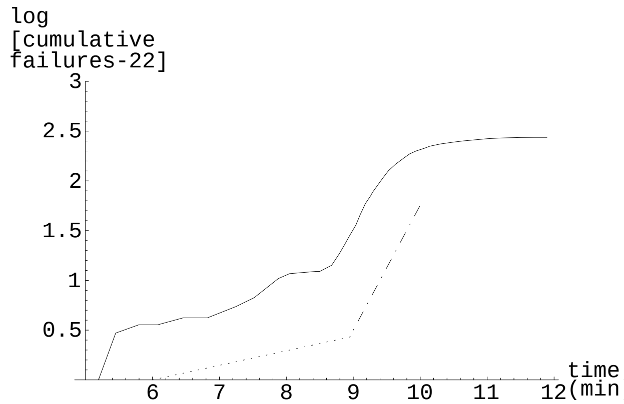


**Figure 6. Log[cumulative line and transformer trips in excess of 22] in August 2003 blackout. The straight line growths correspond to $1.4^{\text{time}}$ and $18^{\text{time}}$ respectively. Time scale is minutes after 16:00 EDT.**

the propagation of the failures. There is an initial number of failures $\theta$. The number of failures at stage $j$ is the random variable $M_j$. The mean number of failures $EM_j$ increases by a factor $\lambda$ in each stage. More precisely,

$$EM_j = \theta\lambda^j \tag{1}$$

The mean cumulative number of failures at stage $j$ is

$$E\sum_{i=0}^{j} M_i = \theta(1+\lambda+\lambda^2+...+\lambda^j) = \theta\frac{\lambda^{j+1}-1}{\lambda-1} \tag{2}$$

The critical case occurs for $\lambda = 1$ [8, 5]. Moreover, if $\lambda > 1$, as $j \to \infty$,

$$M_j\lambda^{-j} \to \theta W \quad \text{a.s.} \tag{3}$$

where $W$ is a random variable with $EW = 1$ that is constant in time. That is, as $j \to \infty$,

$$\log M_j \sim j\lambda + \log(\theta W) \tag{4}$$

To give some examples of this convergence, we simulate the branching process for various values of $\lambda$. This is shown in Figures 7-10. The convergence improves as $\lambda$ increases away from 1.

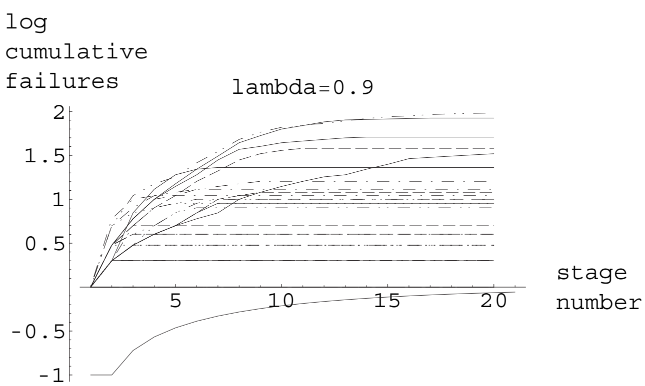


**Figure 7. 40 samples of Galton-Watson branching process for $\lambda = 0.9$. The lower curve is $\lambda^i - 1$ where $i$ is stage number to show the form but not vertical placing of (11).**

The subcritical case of $\lambda = 0.9$ looks quite different from the other figures as shown in Figure 7. The asymptotic slope is zero as the cascade ends. The supercritical case of $\lambda = 1.1$ contains some samples in which the cascade dies out as shown in Figure 8. This is expected and the probability of this can be computed from $\lambda$ as explained in section 4.1. The slightly supercritical cases that die out are qualitatively similar to slightly subcritical cases that die out. However, when we identify an exponentially growing

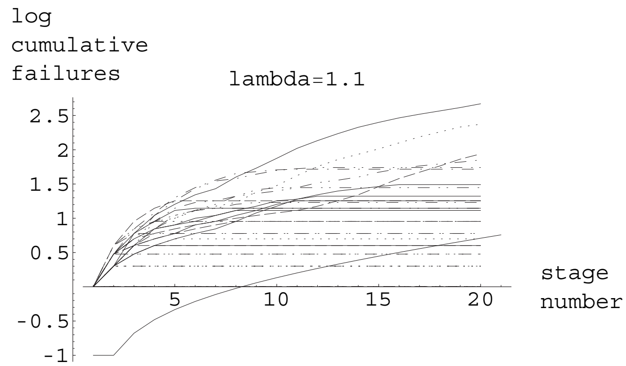


**Figure 8. 40 samples of Galton-Watson branching process for $\lambda = 1.1$. The lower curve is $\lambda^i - 1$ where $i$ is stage number to show the form but not vertical placing of (11).**

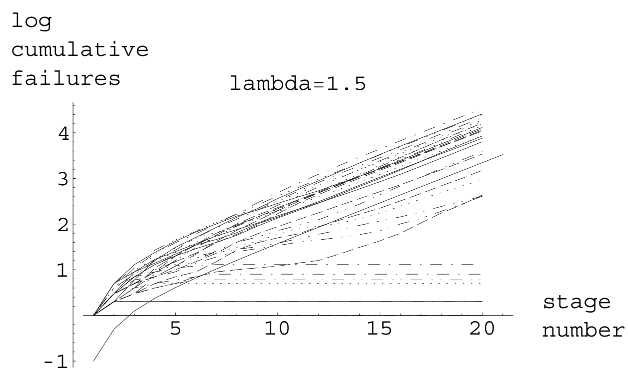


**Figure 9. 40 samples of Galton-Watson branching process for $\lambda = 1.5$. The lower curve is $\lambda^i - 1$ where $i$ is stage number to show the form but not vertical placing of (11).**

phase in blackout data, we already know that the cascade did not die out and we can expect the measured slope on the log plot to reflect the value of $\lambda$.

The discussion so far has not specified the relation of the stages of the Galton-Watson branching process to time and we now outline the first approach to this issue. We suppose that failure data is available that includes the time of each failure and perhaps some additional data explaining the causes of the failure and specifying the type and location of the failure. Then these data used to group the failures into stages. Examples of factors that would tend to group several failures into the same stage could be their closeness in time or location, or being caused by failures in a previous stage. In the initial analysis in this paper we only consider the closeness in time; that is, we group together several failures if they are close in time and neglect the other possible

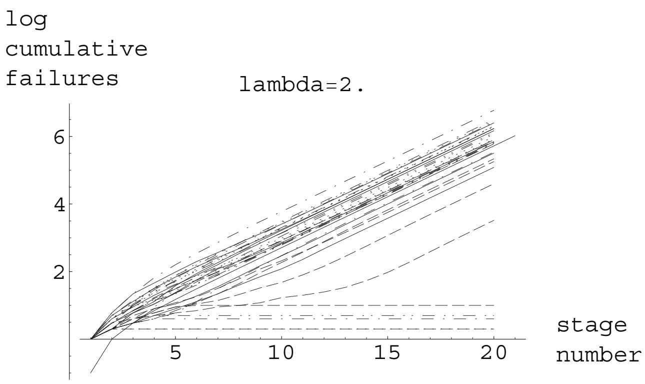


**Figure 10. 40 samples of Galton-Watson branching process. The lower curve is $\lambda^i - 1$ where $i$ is stage number to show the form but not vertical placing of (11).**

factors. In any case one applies criteria to group the failures into stages and then regards the failures in each stage as arising from a Galton-Watson branching process. In this model, there is no attempt to represent the time at which the stages occur. Indeed the series of times near which failures in each stage occur will generally be non-uniformly spaced. That is, one can regard the stages as occurring with a variable time between stages and this timing is not specified within the branching process model in this approach.

### 3.2. Galton-Watson branching process with fixed time between stages

We now discuss the second approach to relating the Galton-Watson branching process to time. This approach groups the failures into stages as in the first approach in section 3.1, but then makes the explicit simplification or approximation that the stages occur with fixed time $b$ between the stages. $b$ is chosen to be the average time between stages and is computed by dividing the time interval $T$ over which the branching process model is applied by the number of stages $J$.

This explicit description of the stage times has several consequences. At each stage of time $b$ minutes, the mean number of failures $EM_j$ increases by a factor $\lambda$ so that the mean number of failures grows exponentially in time with exponent

$$\mu = \ln(\lambda)/b \qquad (5)$$

min$^{-1}$. More precisely, the mean number of failures is $\theta e^{\mu t_j}$ at the stage times $t_j = jb$.

The mean cumulative number of failures at time $jb$ is given by (2). The mean cumulative number of failures is piecewise constant with jumps at each stage and samples of the cumulative number of failures at each stage are asymptotically exponential with exponent $\mu = \ln(\lambda)/b$ min$^{-1}$, the same as the exponent for the mean number of failures.

When fitting this branching process model to failure data, one can fit an exponential $e^{\mu t}$ to a time interval of the data of length $T$ as is done in section 2. This yields an estimate of the number of stages $\hat{J}$ and an estimate of the time between stages $\hat{b} = T/\hat{J}$. Then from (5) we have

$$\hat{\lambda} = e^{\mu\hat{b}} = e^{\mu T/\hat{J}} \qquad (6)$$

One consequence of this approach is that in cases where there are several plausible ways to group the failure data into stages, there can be different estimates $\hat{J}$ of the numbers of stages and hence different estimates $\hat{\lambda}$. A larger number of stages yields a $\hat{\lambda}$ closer to 1. The variation of $\hat{\lambda}$ with the estimated number of stages $\hat{J}$ is expected because $\lambda$ is defined as the expected number of failures per failure in the previous stage and so depends on the stages. In the supercritical case of $\lambda > 1$, increasing the number of stages shortens the time between stages and must decrease the average number of failures that occur over the shorter time between stages. However the supercriticality ($\lambda > 1$) or subcriticality ($\lambda < 1$) is independent of the time between stages.

### 3.3. Continuous time Markov branching process

The third approach to relating the Galton-Watson branching process to time considers a branching process that produces failures at variable intervals in continuous time. One simple assumption is that each failure causes its subsequent failures at a constant rate $1/a$. That is, when each failure occurs, the next failures "caused" by this particular failure will occur at a random time governed by an exponential random variable with parameter $1/a$. The mean time to this next failures is $a$. When these next failures occur, their number is governed by a fixed distribution with mean value $\lambda_c$. For example, the fixed distribution could be a Poisson distribution. The failures existing at any time propagate to cause more failures independently and at different random times. It follows that if there are $M(t)$ failures at time $t$, then the next failures occur after a time interval governed by an exponential random variable with parameter $M(t)/a$. This is a standard one dimensional continuous time Markov branching process [1]. Write $Z(t)$ for the number of failures at time $t$ and $\theta$ for the initial number of failures at time zero. The mean number of failures is exponential:

$$EZ(t) = \theta e^{\mu t} \qquad (7)$$

where

$$\mu = (\lambda_c - 1)/a \qquad (8)$$

Moreover, if $\mu > 0$, as $t \to \infty$,

$$Z(t)e^{-\mu t} \to \theta W \quad \text{a.s.} \tag{9}$$

where $W$ is a random variable with $EW = 1$ that is constant in time. That is, as $t \to \infty$,

$$\log Z(t) \sim \mu t + \log(\theta W) \tag{10}$$

(Sampling $Z(t)$ at regular intervals $\delta$ of time yields $Z(0)$, $Z(\delta)$, $Z(2\delta)$, $Z(3\delta)$,... and this is a Galton-Watson branching process. However, one does not necessarily recover the original Galton-Watson branching process by this sampling. For example, a Galton-Watson branching process produced with a Poisson distribution is not embeddable in any continuous time Markov branching process [1] and so cannot be the sampled Galton-Watson process.)

It follows from (7) that the mean cumulative number of failures is

$$E\int_0^t Z(\tau)d\tau = \frac{\theta}{\mu}\left(e^{\mu t} - 1\right) \tag{11}$$

If $\mu > 0$, it follows from (9) that

$$\int_0^t Z(\tau)d\tau \sim \frac{\theta}{\mu}\left(e^{\mu t} - 1\right)W \tag{12}$$

and, as $t \to \infty$,

$$\log \int_0^t Z(\tau)d\tau \sim \mu t + \log(\theta W/\mu) \tag{13}$$

so that plotting $\log \int_0^t Z(\tau)d\tau$ against $t$ gives an asymptotic slope of $\mu$. This result supports the procedure in section 2 as long as convergence near to the asymptotic slope is achieved before saturation effects apply.

Examining the cumulative number of failures as a function of time avoids much of the difficulties of grouping blackout data into stages. That is, this approach is largely insensitive to how previous failures were grouped, it only needs to know that they happened in the past.

For the continuous time Markov branching process model, the process evolves in jumps. (See [1], where the jumps are also referred to as splits. Note that the jumps in this process do not correspond to the stages of the fixed stage Galton-Watson branching process model.) At each jump, one of the previous failures is replaced by an average of $\lambda_c$ failures so that the average number of failures increases by $\lambda_c - 1$. Write $S_1$, $S_2$, $S_3$, ... for the number of failures at jumps $1, 2, 3$, ... Then the increments in the $S_r$ are independent, identically distributed random variables with mean $\lambda_c - 1$. Under suitable conditions assuring that the considered cascades do not die out as detailed in [1],

$$\frac{S_r}{r} \to \lambda_c - 1 \qquad \text{as} \quad r \to \infty \tag{14}$$

This motivates us to group the more nearly simultaneous failures in the exponential increasing phase into jumps to obtain $S_1$, $S_2$, $S_3$, ..., and to examine $S_1/1$, $S_2/2$, $S_3/3$, ... for any indication of convergence to $\lambda_c - 1$.

### 3.4. Fitting branching models to the blackout data

One can readily conclude that both a supercritical fixed stage Galton-Watson branching process and a supercritical continuous time Markov branching process model are consistent with the exponentially increasing phases of the blackout data in section 2. This conclusion is insensitive to the generating function of the branching process. The criticality for both processes occurs at $\lambda = 1$ or $\lambda_c = 1$ (in the case of the continuous time Markov branching process model $\lambda_c = 1$ corresponds to $\mu = 0$ according to (8)). (Otter's theorem [8] shows that the power tail in the distribution of total number of failures occurs at $\lambda = 1$ for generic assumptions on the generating function.)

To progress beyond this qualitative modeling of the exponential blackout phases as a supercritical branching process, we need to estimate model parameters. Since the WSCC August 1996 blackout has very sparse data and the raw data for the August 2003 blackout is not yet available for study, we illustrate estimating model parameters for the WSCC July 1996 blackout using the discrete and continuous time branching process models. The time period of the exponential growth is chosen to be the 7 minutes from 14:24 to 14:31 MDT. Section 2 fit the exponent of the exponential growth in this time period as $\mu = 0.47$ min$^{-1}$. The failure times are shown in Figure 11 and Table 1.

For the Galton-Watson branching process models, we group the failures into stages according to their closeness in time. Successive failures are grouped into the same stage if the time between them is less than a fraction $\delta$ of the average time between failures. For illustration we choose $\delta = 0.5$. The average time between failures for the failure times in Figure 11 is $0.42$ min so that (average time between failures)$\delta = 0.21$ and the corresponding grouping into 8 stages is indicated in the third column of Figure 11.

In the first modeling approach of section 3.1, we plot the logarithm of the cumulative staged failures as a function of stage number and fit these data with an exponential as shown in Figure 12 to obtain $\hat{\lambda} = 1.4$.

In the second modeling approach of section 3.2, the estimated number of stages is $\hat{J} = 8$. Then the estimated stage time $\hat{b} = T/\hat{J} = 7/\hat{J} = 0.875$ and (6) gives $\hat{\lambda} = 1.5$. As discussed in section 3.2, a different assumption about the grouping into stages would give different estimates. For example, choosing $\delta = 1$ would result in fewer stages so that $\hat{J} = 5$, $\hat{b} = 1.2$ and $\hat{\lambda} = 1.9$.

In the third, continuous time modeling approach of section 3.3, we need to group the failure data into jumps.

Although the jumps in the continuous time process do not directly correspond to the stages of the Galton-Watson branching process, we can for illustrative purposes use the same grouping of failure data into jumps as used for stages in Table 1. The consequent evolution of the number of failures and the series (14) is shown in Table 2. The series in Table 2 is too short and noisy for convergence to be verified, but the average of the last four elements estimates the limit of series (14) as $\approx 0.5$ and this yields $\hat{\lambda}_c = 1.5$. This implies using $\mu = 0.47$ and (8) that the average time for one failure to split is $\hat{a} = 1.1$.

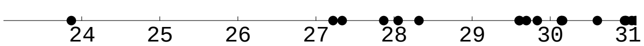


**Figure 11. Times of line trips in WSCC July 1996 blackout in minutes after 14:00 MDT.**

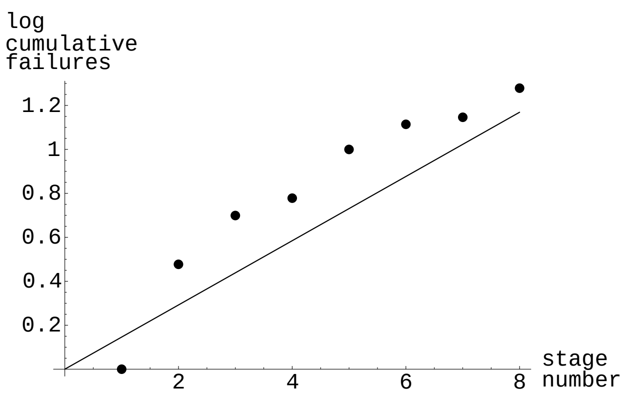


**Figure 12. Log cumulative failures in exponential phase of WSCC July 1996 blackout as a function of stage number. The straight line is an exponential with exponent 1.4.**

We do not yet have evidence available that the continuous time Markov branching process approximates the time sequence of actual failures; the present argument in favor of this modeling is that the assumptions are simple. Given a longer time series of failure data, such as the failure times in the August 2003 blackout, we could try to discriminate between the models and approaches suggested here and statistically or qualitatively test the fit of the models to the data. The data currently available to us is too limited to attempt this.

**Table 1. Line trip times and stage numbers for exponential phase of July 1996 WSCC blackout. Time units are minutes after 14:00 MDT.**

| trip time | increment in trip time | stage number |
|---|---|---|
| 23.867 | | 1 |
| 27.219 | 3.352 | 2 |
| 27.336 | 0.117 | 2 |
| 27.868 | 0.532 | 3 |
| 28.052 | 0.184 | 3 |
| 28.319 | 0.267 | 4 |
| 29.602 | 1.283 | 5 |
| 29.608 | 0.006 | 5 |
| 29.695 | 0.087 | 5 |
| 29.835 | 0.140 | 5 |
| 30.144 | 0.309 | 6 |
| 30.145 | 0.001 | 6 |
| 30.159 | 0.014 | 6 |
| 30.604 | 0.445 | 7 |
| 30.953 | 0.349 | 8 |
| 30.965 | 0.012 | 8 |
| 30.971 | 0.006 | 8 |
| 31.045 | 0.074 | 8 |
| 31.094 | 0.049 | 8 |
| 31.815 | 0.721 | |

**Table 2. Evolution of number of failures in jumps**

| $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $S_r$ | 1 | 2 | 2 | 1 | 4 | 3 | 1 | 5 |
| $S_r/r$ | 1 | 1 | 0.67 | 0.25 | 0.8 | 0.5 | 0.14 | 0.63 |

## 4. Implications of branching model

Now we suppose that blackouts can be approximated by a discrete time Galton-Watson branching process model and explore some illustrative calculations using the model.

### 4.1. Probability of a given large blackout not happening

One interesting exercise is to use values of $\lambda$ estimated from the large real blackouts that occurred with a region of exponential increase to compute the probability $q$ of those blackouts *not* having the region of exponential increase. Of course there may be a blackout without the region of exponential increase, but such a blackout will have much more limited size. Thus we consider $q$ to be the probability that the cascade dies out for that given value of $\lambda$. (This was alluded to in the discussion of Figure 8 above.) The value of $q$ is the same for the Galton-Watson branching process and for the continuous time Markov branching process, but it does depend on the generating function $f(s)$ that is used to construct these branching processes. Here we will assume that the generating function corresponds to a Poisson distribution so that $f(s) = e^{\lambda(s-1)}$, as suggested by the branching process approximation to the abstract cascading failure model in [7, 5]. The probability $q$ is easily computed as a root of the equation $f(s) = s$ and the results are shown in Table 3.

Suppose that a blackout has an exponential phase with $\lambda \approx 2$. This would imply that the probability that the exponential phase of the blackout did not occur is about 0.2. This calculation is made in hindsight after the blackout, but it does highlight the difficulties of making optimal decisions during the evolution of the blackout, even given good information. Blackouts with lower values of $\lambda$ will have higher values of $q$. Suppose that $\lambda = 1.1$ and an exponentially increasing blackout occurred. The probability that it did not occur is $0.82$ and one could argue that, in the absence of real time information about risk, a competent and well-informed system operator might well have acted properly by assuming the most likely outcome of no large blackout.

**Table 3. Probability $q$ of large blackout not occurring**

| $\lambda$ | $q$ |
|---|---|
| 0.9 | 1.00 |
| 1.0 | 1.00 |
| 1.1 | 0.82 |
| 1.2 | 0.69 |
| 1.3 | 0.58 |
| 1.4 | 0.49 |
| 1.5 | 0.42 |
| 1.6 | 0.36 |
| 1.7 | 0.31 |
| 1.8 | 0.27 |
| 1.9 | 0.23 |
| 2.0 | 0.20 |
| 3.0 | 0.06 |
| 4.0 | 0.02 |
| 5.0 | 0.01 |

## 4.2. An initial approach to real time monitoring of cascading blackouts

The exponential cascading phase starts slowly and accelerates later. As we accumulate more failures, the probability of an exponentially accelerating cascade increases. Is it possible to detect this increased probability during the slow part? This subsection outlines an approach to quantify the statistics of this problem by monitoring cumulative line failures. Monitoring cumulative line failures would be practical in real time in a well-instrumented control center.

We regard the $\lambda$ parameter of a staged branching process as a random variable $\Lambda$. Let the probability density function of $\Lambda$ for a given system condition (i.e. stress level) be $f_\Lambda(\lambda)$ in $[\lambda_{\min}, \lambda_{\max}]$. We assume that for given system condition we have either historical data or off-line simulations giving $f_\Lambda(\lambda)$.

We have set a threshold to limit cascading failure risk of $\lambda < \lambda_t$. Presumably $\lambda_t < 1$ to exclude the possibility of exponentially increasing phases when the power system is operated with $\lambda < \lambda_t$.

Let the cumulative line failures observed in real time be $S$. Note that it would be necessary to somehow distinguish line failures involved in the initial disturbance from those involved in the cascading phase.

Suppose that we observe in real time that $S = k$. Then we know that $S \geq k$ for the final cascade. So what, knowing that $S \geq k$, is the probability of cascading failure caused by $\lambda > \lambda_t$? That is, what is $P[\Lambda > \lambda_t | S \geq k]$? We give a sample calculation of $P[\Lambda > \lambda_t | S \geq k]$ below. In particular, we numerically evaluate $P[\Lambda > \lambda_t | S \geq k]$ under some assumptions for increasing values of $k$. This quantifies how the probability of a large cascade increases as the number of observed line trips increases.

The calculations are done by evaluating the formula (27) derived in the appendix. The assumptions are that the cascade is modeled by a Galton-Watson branching process generated by a Poisson distribution. The distribution of $\Lambda$ on $[\lambda_{\min}, \lambda_{\max}]$, in the absence of any information about the likely form of this distribution, is assumed to be uniform. Saturation effects are neglected.

The data needed is the initial number of failures $\theta$ and the range $[\lambda_{\min}, \lambda_{\max}]$ for the uniform distribution of $\Lambda$. We choose values of these parameters for a sample calculation and vary $k$. The results are shown in Table 4. When $k = 1$ there is no information supplied by the line trip that is additional to the information that a cascade has started (the probability of $\lambda > 0.9$ is clearly $0.5$ when $\lambda$ is uniformly distributed in $[0.7, 1.1]$. The probability of $\lambda > 0.9$ increases with $k$, but the rate of increase is modest in this example. Thus in this example the real time monitoring of $k$ would add little value to the offline calculation of the distribution of $\Lambda$. Note that waiting until large $k$ is observed does not help manage the cascading blackout because then the cascading process is well under way and cannot readily be corrected.

**Table 4. Probability of $\lambda$ exceeding threshold $\lambda_t$ when $k$ line trips are observed.**

| $k$ | $P[\Lambda > \lambda_t \vert S \geq k]$ | parameters |
|---|---|---|
| 1 | 0.50 | $\lambda_t = 0.9$ |
| 2 | 0.52 | $[\lambda_{\min}, \lambda_{\max}] = [0.7, 1.1]$ |
| 3 | 0.53 | $\theta = 1$ |
| 4 | 0.55 | |
| 5 | 0.57 | |
| 6 | 0.58 | |
| 7 | 0.60 | |
| 8 | 0.61 | |
| 9 | 0.62 | |
| 10 | 0.63 | |
| 15 | 0.69 | |
| 20 | 0.73 | |

## 5. Conclusion

The main contribution of this paper is to observe in recent North American cascading failure blackouts exponentially increasing phases of cumulative line trips and suggest that these be modeled by supercritical Markov branching processes. Simple discrete time branching process models and a continuous time Markov branching process model are considered. Several initial calculations illustrating how parameters may be estimated and these models might be applied are suggested. One interesting consequence of this statistical blackout modeling is that the probability of a given blackout *not* occurring could be estimated after the blackout. A preliminary approach to real time blackout monitoring is considered.

The blackout data sets currently available to us are not long enough to distinguish between the models or definitively estimate the parameters, but, when they are supercritical, all the branching process models do qualitatively reproduce the exponential growth of failures that seems to be the manner in which the 1996 and 2003 North American blackouts became widespread.

## 6. Acknowledgements

I. Dobson and B.A. Carreras gratefully acknowledge coordination of this work by the Consortium for Electric Reliability Technology Solutions and funding in part by the Assistant Secretary for Energy Efficiency and Renewable Energy, Office of Power Technologies, Transmission Reliability Program of the U.S. Department of Energy under contract 9908935 and Interagency Agreement DE-A1099EE35075 with the National Science Foundation. I. Dobson, D.E. Newman, and B.A. Carreras gratefully acknowledge support in part from NSF grants ECS-0214369 and ECS-0216053. Part of this research has been carried out at Oak Ridge National Laboratory, managed by UT-Battelle, LLC, for the U.S. Department of Energy under contract number DE-AC05-00OR22725.

## References

[1] K.B. Athreya, P.E. Ney, *Branching Processes*, Dover NY 2004 (reprint of Springer-verlag Berlin 1972).

[2] B.A. Carreras, V.E. Lynch, I. Dobson, D.E. Newman, Critical points and transitions in an electric power transmission model for cascading failure blackouts, *Chaos*, vol. 12, no. 4, December 2002, pp. 985-994. *Note: all the authors' papers are available at* http://eceserv0.ece.wisc.edu/∼dobson/home.html

[3] B.A. Carreras, D.E. Newman, I. Dobson, A.B. Poole, Evidence for self-organized criticality in a time series of electric power system blackouts, to appear (accepted March 2004) in *IEEE Transactions on Circuits and Systems, Part 1*.

[4] I. Dobson, B.A. Carreras, D.E. Newman, A probabilistic loading-dependent model of cascading failure and possible implications for blackouts, *36th Hawaii International Conference on System Sciences*, Hawaii, January 2003.

[5] I. Dobson, B.A. Carreras, D.E. Newman, A branching process approximation to cascading load-dependent system failure. *37th Hawaii International Conference on System Sciences*, Hawaii, January 2004.

[6] I. Dobson, B.A. Carreras, D.E. Newman, Probabilistic load-dependent cascading failure with limited component interactions, *IEEE International Symposium on Circuits and Systems*, Vancouver Canada, May 2004.

[7] I. Dobson, B.A. Carreras, D.E. Newman, A loading-dependent model of probabilistic cascading failure, to appear in *Probability in the Engineering and Informational Sciences*, vol. 19, no. 1, 2005.

[8] T.E. Harris, *Theory of branching processes*, Dover NY 1989.

[9] NERC (North American Electric Reliability Council) (2002). *1996 system disturbances*. (Available from NERC, Princeton Forrestal Village, 116-390 Village Boulevard, Princeton, New Jersey 08540-5731.)

[10] U.S.-Canada Power System Outage Task Force, *Final Report on the August 14th blackout in the United*

*States and Canada*. United States Department of Energy and National Resources Canada, April 2004.

[11] V. Venkatasubramanian, Y. Li, Analysis of 1996 Western American Electric Blackouts, *Bulk Power System Dynamics and Control - VI*, Cortina d'Ampezzo, Italy, August 2004.

## A. Probability of $\lambda > \lambda_t$ given $k$ failures

For a Galton-Watson branching process with a finite number of components, the probability distribution of total number of failures $S$ for a given value of $\lambda$ is $P[S = s|\Lambda = \lambda]$ given by saturating generalized Poisson distribution [5].

$$g(r, \theta, \lambda, n) = \theta(r\lambda + \theta)^{r-1}\frac{e^{-r\lambda-\theta}}{r!} \quad ; \quad 0 \le r \le (n-\theta)/\lambda,\ r < n \tag{15}$$

$$g(r, \theta, \lambda, n) = 0\ ; \quad (n-\theta)/\lambda < r < n,\ r \ge 0 \tag{16}$$

$$g(n, \theta, \lambda, n) = 1 - \sum_{s=0}^{n-1} g(s, \theta, \lambda, n) \tag{17}$$

Then joint distribution of $(S, \Lambda)$ is

$$f_{S,\Lambda}(s, \lambda) = P[S = s|\Lambda = \lambda] f_\Lambda(\lambda) \tag{18}$$

$$P[\Lambda > \lambda_t | S \ge k] = \frac{P[\Lambda > \lambda_t \text{ and } S \ge k]}{P[S \ge k]} \tag{19}$$

$$= \frac{\sum_{s=k}^{n} \int_{\lambda_t}^{\lambda_{\max}} f_{S,\Lambda}(s, \lambda)\, d\lambda}{\sum_{s=k}^{n} \int_{\lambda_{\min}}^{\lambda_{\max}} f_{S,\Lambda}(s, \lambda)\, d\lambda} \tag{20}$$

$$= \frac{\int_{\lambda_t}^{\lambda_{\max}} P[S \ge k|\Lambda = \lambda] f_\Lambda(\lambda)\, d\lambda}{\int_{\lambda_{\min}}^{\lambda_{\max}} P[S \ge k|\Lambda = \lambda] f_\Lambda(\lambda)\, d\lambda} \tag{21}$$

Since $\Lambda$ is assumed to be uniformly distributed on $[\lambda_{\min}, \lambda_{\max}]$,

$$P[\Lambda > \lambda_t | S \ge k] = \frac{\int_{\lambda_t}^{\lambda_{\max}} P[S \ge k|\Lambda = \lambda]\, d\lambda}{\int_{\lambda_{\min}}^{\lambda_{\max}} P[S \ge k|\Lambda = \lambda]\, d\lambda} \tag{22}$$

$$= \frac{\int_{\lambda_t}^{\lambda_{\max}} 1 - F_{S|\Lambda=\lambda}(k-1)\, d\lambda}{\int_{\lambda_{\min}}^{\lambda_{\max}} 1 - F_{S|\Lambda=\lambda}(k-1)\, d\lambda} \tag{23}$$

$$= \frac{\lambda_{\max} - \lambda_t - \sum_{s=0}^{k-1} \int_{\lambda_t}^{\lambda_{\max}} g(s, \theta, \lambda, n)\, d\lambda}{\lambda_{\max} - \lambda_{\min} - \sum_{s=0}^{k-1} \int_{\lambda_{\min}}^{\lambda_{\max}} g(s, \theta, \lambda, n)\, d\lambda} \tag{24}$$

$$= \frac{(\lambda_{\max} - \lambda_t)(1 - e^{-\theta}) - \sum_{s=1}^{k-1} \int_{\lambda_t}^{\lambda_{\max}} g(s, \theta, \lambda, n)\, d\lambda}{(\lambda_{\max} - \lambda_{\min})(1 - e^{-\theta}) - \sum_{s=1}^{k-1} \int_{\lambda_{\min}}^{\lambda_{\max}} g(s, \theta, \lambda, n)\, d\lambda} \tag{25}$$

Suppose that $k-1$ is small enough to avoid saturation effects. Then

$$\int_{\lambda_t}^{\lambda_{\max}} g(s, \theta, \lambda, n)\, d\lambda = -\frac{\theta}{ss!}\Gamma[s, \theta + s\lambda]\Big]_{\lambda_t}^{\lambda_{\max}} \tag{26}$$

and

$$P[\Lambda > \lambda_t | S \ge k] = \frac{(\lambda_{\max} - \lambda_t)(1 - e^{-\theta}) - \sum_{s=1}^{k-1} \frac{\theta}{ss!}\Gamma[s, \theta + s\lambda]\Big]_{\lambda_{\max}}^{\lambda_t}}{(\lambda_{\max} - \lambda_{\min})(1 - e^{-\theta}) - \sum_{s=1}^{k-1} \frac{\theta}{ss!}\Gamma[s, \theta + s\lambda]\Big]_{\lambda_{\max}}^{\lambda_{\min}}} \tag{27}$$