

## A branching process approximation to cascading load-dependent system failure

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### Abstract

*Networked infrastructures operated under highly loaded conditions are vulnerable to catastrophic cascading failures. For example, electric power transmission systems must be designed and operated to reduce the risk of widespread blackouts caused by cascading failure. There is a need for analytically tractable models to understand and quantify the risks of cascading failure. We study a probabilistic model of loading dependent cascading failure by approximating the propagation of failures as a Poisson branching process. This leads to a criticality condition for the failure propagation. At criticality there are power tails in the probability distribution of cascade sizes and consequently considerable risks of widespread catastrophic failure. Avoiding criticality or supercriticality is a key approach to reduce this risk. This approach of minimizing the propagation of failure after the cascade has started is complementary to the usual approach of minimizing the risk of the first few cascading failures. The analysis introduces a saturating form of the generalized Poisson distribution so that supercritical systems with a high probability of total failure can be considered.*

### 1. Introduction

Networked infrastructures such as electric power transmission systems are vulnerable to widespread cascading failures when the systems are highly loaded. Since modern society depends on large infrastructures, catastrophes in which failures propagate to most or all of the system are of concern. For example, blackouts of substantial portions of the North American power system east or west of the Rocky Mountains have a huge cost to society, as demonstrated in 2003 and 1996 respectively. There is a need for analytically tractable models to understand and quantify the risks of cascading failure so that networked systems can be designed and operated to reduce the risk of catastrophic failure.

Analyses of 15 years of North American blackout data

show an empirical probability distribution of blackout size which has heavy tails and evidence of power law dependence in these tails [24, 2, 11, 3, 6]. The exponent of the power tail is roughly estimated to be in the range  $-2$  to  $-1$ . These data show that large blackouts are much more likely than might be expected from a distribution of blackout size in which the tails decay exponentially. Simulation models of cascading blackouts show similar power tails and the power tails have been attributed to the nature of the cascading process [19, 9, 7].

Because of protection and appropriate design and operational procedures, it is very rare for power transmission components to fail in the sense of the component breaking. However, it is routine for these components to be temporarily removed from service by protection equipment and the outaged or tripped component is then failed in the sense that it is temporarily not available to transmit power. Moreover there are sometimes misoperations or mistakes in protection, communication and control systems or operational procedures or sometimes the power system is operated under conditions that could not be anticipated in the original design settings or procedures. In the context of power transmission systems, the term “failure” as used in this paper should be understood in this broad and nuanced sense.

Notable general features of power transmission systems are the large number of components, the increased probability of component failure and interaction at high load, and the numerous, varied and widespread interactions between components. Large blackouts typically involve long sequences of component failures. Many of the interactions are rare, unanticipated or unusual, not least because of engineering efforts to design and operate the system so as to avoid the most common failures and interactions. Although we use electric power transmission system blackouts as the motivating example in this paper, these general features appear in other networked infrastructures so that it is likely that the ideas apply more generally.

One natural way to study cascading failure is to consider the failures propagating probabilistically according to a Galton-Watson-Bienaymé branching process [23]. For example, simple assumptions lead to a Poisson branching

process that has the total number of components failed distributed according to the generalized Poisson distribution [17, 15].

On the other hand, the CASCADE model of probabilistic cascading failure [20] has the following general features:

1. Multiple identical components, each of which has a random initial load and an initial disturbance.
2. When a component overloads, it fails and transfers some load to the other components.

Property 2 can cause cascading failure: a failure additionally loads other components and some of these other components may also fail, leading to a cascade of failure. The components become progressively more loaded and the system becomes weaker as the cascade proceeds.

Both the Poisson branching process and CASCADE can exhibit criticality and power tails in the probability distribution of the number of failed components.

We begin the paper by reviewing standard results on branching processes and the generalized Poisson distribution and then consider the implications of these results for the risk of load-dependent cascading failure. A saturating form of the generalized Poisson distribution is introduced to allow study of the transition through criticality in a system with a large but finite number of components. We review the CASCADE model of cascading failure and then show how CASCADE can be approximated by the saturating generalized Poisson distribution. Then we discuss the implications of the approximation for analyzing CASCADE and understanding cascading failure in blackouts.

## 2. Review of branching processes

This section reviews standard material on Galton-Watson-Bienaymé branching processes [23] and generalized Poisson distributions [17, 15] as expressed in terms of cascading failures.

### 2.1. Generalities

We first consider an infinite number of system components. All components are initially unfailed. Component failures occur in stages with  $M_i$  the number of failures in stage  $i$ . We first assume an initial disturbance that causes one failure in stage zero so that  $M_0 = 1$ . This first failure is considered to cause a certain number of failures  $M_1$  in stage 1.  $M_1$  is determined according to a probability distribution with generating function  $E[t^{M_1}] = f(t)$  and mean  $\lambda$ . In subsequent stages, each of the  $M_i$  failures in stage  $i$  independently causes a further number of failures in stage  $i + 1$  according to the same distribution  $f(s)$ . That is, the

$k$ th failure in stage  $i$  causes  $M_{i+1}^{(k)}$  failures in stage  $i + 1$  and

$$M_{i+1} = M_{i+1}^{(1)} + M_{i+1}^{(2)} + \dots + M_{i+1}^{(M_i)} \quad (1)$$

where  $M_{i+1}^{(1)}, M_{i+1}^{(2)}, \dots, M_{i+1}^{(M_i)}$  are independent. This independence is a plausible approximation in a system with many components and many component interactions so that series of failures propagating in parallel can be assumed not to interact. The generating function of  $M_k$  is

$$E[t^{M_k}] = f(f(f(\dots f(t)\dots))) = f^{(k)}(t) \quad (2)$$

and the mean  $E[M_k] = \lambda^k$ . If at any stage  $k$ ,  $M_k = 0$ , then zero elements fail for all subsequent stages and the cascading process terminates.

There are three cases, depending on the mean  $\lambda$  of the number of failures caused by each failure in the previous stage. In the subcritical case  $\lambda < 1$ , a finite number of components will fail. In the supercritical case  $\lambda > 1$ , either a finite or infinite number of components can fail and the number of failures in each stage tends to zero or infinity respectively. The critical case is  $\lambda = 1$ .

We are most interested in the distribution of the total number of failures

$$M = \sum_{k=0}^{\infty} M_k \quad (3)$$

The generating function of  $M$  is  $F(t) = E[t^M]$  and it satisfies the recursion  $F(t) = tf(F(t))$ .

### 2.2. Universality of the critical exponent

Under mild conditions on  $f$ , for the critical case  $\lambda = 1$ ,  $P[M = r] \sim r^{-\frac{3}{2}}$  as  $r \rightarrow \infty$  [26, 23]. That is, the distribution of the total number of failures of a branching process at criticality has a universal property of a power tail with exponent  $-\frac{3}{2}$ . The details are in Otter's theorem [26]:

**Theorem 1** *Suppose that  $P[M_1 = 0] > 0$  and that there is a point  $a$  in the interior of the circle of convergence of  $f$  for which  $f'(a) = f(a)/a$ . (This is true, for example, if  $1 < \lambda \leq \infty$  or if  $f(s)$  is entire or if  $f'(\rho) = \infty$ , where  $\rho$  is the radius of convergence of  $f$ . The point  $(a, f(a))$  is then the point where the graph of  $f$ , for real positive  $s$ , is tangent to a line through the origin. Let  $\alpha = a/f(a)$  and let  $d$  be the largest integer such that  $P[M_1 = r] \neq 0$  implies that  $r$  is a multiple of  $d$ ,  $r = 1, 2, \dots$ . If  $r - 1$  is not divisible by  $d$ , then  $P[M = r] = 0$ , while if  $r - 1$  is divisible by  $d$ , then*

$$P[M = r] = d \left( \frac{a}{2\pi\alpha f''(a)} \right)^{\frac{1}{2}} \alpha^{-r} r^{-\frac{3}{2}} + O\left(\alpha^{-r} r^{-\frac{5}{2}}\right) \quad r \rightarrow \infty \quad (4)$$

*Notice that  $\alpha \geq 1$ , the equality holding if and only if  $\lambda = 1$ . Also  $d = 1$  when  $P[M_1 = r] \neq 0$  for  $r = 1, 2, \dots$*

### 2.3. Branching generated by a Poisson distribution

If, in addition to the independence assumptions above, the failures propagate in a large number of components so that each failure has a small uniform probability of independently causing each failure in a large number of other components, then the distribution of failures caused by each failure in the previous stage can be approximated as a Poisson distribution [17] so that

$$P[M_1 = m] = \frac{\lambda^m}{m!} e^{-\lambda}, \quad m = 0, 1, 2, \dots \quad (5)$$

$$f(t) = e^{\lambda(t-1)} \quad (6)$$

The distribution of the total number of failures becomes

$$P[M = r] = (r\lambda)^{r-1} \frac{e^{-r\lambda}}{r!}, \quad 0 \leq \lambda \leq 1 \quad (7)$$

which is known as the Borel distribution.

### 2.4. A probabilistic initial disturbance and the generalized Poisson distribution

If we neglect the zero stage that has one failure, and consider the failures starting with stage 1, then (5) gives a distribution of initial failures according to a Poisson distribution with mean  $\lambda$ .

However, we distinguish the initial failures that are caused by some initial disturbance from the subsequent propagation of failures internal to the system. We want to represent the initial disturbance by its own probability distribution. This can be done by specifying a probability distribution for  $M_0$ , the number of failures in stage zero. If the initial failures are Poisson distributed with mean  $\theta$  so that

$$P[M_0 = m] = \frac{\theta^m}{m!} e^{-\theta}, \quad m = 0, 1, 2, \dots \quad (8)$$

$$f_0(t) = e^{\theta(t-1)} \quad (9)$$

then the generating function of  $M_k$  becomes  $f_0(f^{(k)}(t))$  and the distribution of the total number of failures becomes

$$P[M = r] = \theta(r\lambda + \theta)^{r-1} \frac{e^{-r\lambda - \theta}}{r!}, \quad \theta \geq 0, \quad 0 \leq \lambda \leq 1 \quad (10)$$

which is the generalized (or Lagrangian) Poisson distribution introduced by Consul and Jain [17, 12, 15]. The probability generating function of (10) is

$$E[s^M] = e^{\theta(t-1)} \quad \text{where } t \text{ is the function of } s \text{ satisfying} \\ t = se^{\lambda(t-1)} \quad (11)$$

The mean of the generalized Poisson distribution (10) is

$$E[M] = \frac{\theta}{1 - \lambda} \quad (12)$$

The generalized Poisson distribution is usually restricted to parameters such that  $\lambda \leq 1$  to avoid the supercritical case in which there is a finite probability of  $M$  infinite.

### 3. Implications for risk of load-dependent cascading failure

The following sections show how a model of loading dependent cascading failure can be approximated as a branching process. To motivate this topic, this section supposes that cascading failure can be treated as a branching process and discusses some general implications of the branching results in Section 2 for risk analysis and mitigation of cascading failure.

Suppose that the system is at criticality ( $\lambda = 1$ ) so that the probability distribution of the total number of failures  $M$  follows a power law with exponent  $-\frac{3}{2}$ . Since risk  $R$  is the product of probability and cost,

$$R(m) = P[M = m]C[m] \sim m^{-\frac{3}{2}}C[m] \quad (13)$$

First assume in (13) that the cost  $C(m)$  is proportional to the total number of failures  $m$ . (This is a conservative estimate in applications such as blackouts; even if the direct costs are proportional to the blackout size and the total number of failures, the indirect costs can be very high for large blackouts [1].) Then  $R(m) \sim m^{-\frac{3}{2}}m = m^{-\frac{1}{2}}$ . This gives a weak decrease in risk as the number of failures increase, which means that the risk of cascading failure includes a strong contribution from large cascades. Moreover, if instead cost increases according to  $C[m] \sim m^\alpha$  where  $\alpha > \frac{3}{2}$ , then (13) implies that the risk of large cascades exceeds that of small cascades, despite the large cascades being rarer.

Consider a general load dependence for component failure and interaction. We assume that system components are more likely to fail and more likely to cause other component failures when load increases. It is reasonable to assume that at zero load  $\lambda < 1$ , since a system design with a significant risk of cascading failure at zero load is unlikely to be feasible when operated at normal loads. Moreover, if the system is operated at an absurdly high load at which all components are at their limits, then failure of any component will on average cause many other components to fail and then  $\lambda > 1$ . We may also assume that  $\lambda$  is an increasing and continuous function of load. Then there is a critical load for which  $\lambda = 1$  and the branching process is critical and the risk is governed by (13). The risk will be even higher for  $\lambda > 1$ .

Thus a simple criterion for avoiding the high risk of cascading failure associated with  $\lambda \geq 1$  with some margin determined by a choice of  $\lambda_{\max} < 1$  is

$$\boxed{\text{design and operate system so that } \lambda \leq \lambda_{\max} < 1} \quad (14)$$

Although this is a simple criterion, translating it to applicable design and operational criteria is a substantial task. Moreover, applying the criteria (14) generally requires the system to be operated with limited throughput. For example, in electric power transmission systems, the loading of transmission lines and other system components would be limited. Thus limiting the risk of cascading failure using (14) will have an economic cost. The dynamics and difficulties of managing this tradeoff should not be neglected.

One approach to limiting cascading failure is to describe the most likely sequences of cascading failures starting from the initiating failures and design and operate the system to reduce their probability. This standard approach is sensible and can reduce risk [22, 25, 10]. However, in large interconnected and interdependent systems there is a combinatorial explosion of possibilities. It is often impractical to envisage and to quantify and compute probabilities for all but the most likely or apparent of these cascading sequences. A large number of rare and hard to anticipate interactions may have to be neglected [27].

Criterion (14) suggests a different and complementary approach that focusses on limiting the average propagation of failures after a cascade is started.  $\lambda$  is the expected number of failures consequent upon a single failure. We suggest that estimation of average values of  $\lambda$  may be feasible using simulation [8] or otherwise and that the dependence of  $\lambda$  on load and system design could be determined to allow (14) to be implemented. Perhaps the simplifications in this approach could allow the contributions to  $\lambda$  from numerous but rare interactions to be accounted for more readily. There are a number of problems in establishing this approach. Two of these problems are

1. Branching processes usually assume an infinite number of components so that there can be an infinite number of failures in the supercritical case. This is not realistic when considering the transition from subcritical to supercritical.
2. Can loading dependent cascading failure be well approximated as a branching process?

Section 4 addresses problem 1 with a saturating branching process and the rest of this paper addresses problem 2 by showing how the CASCADE model of load-dependent cascading failure can be approximated by the saturating branching process.

#### 4. Saturation due to finite system size

In our application we have a large but finite number  $n$  of components and we need to introduce a saturation or truncation of the Poisson branching process. Let

$$N = \min\{n - 1, \text{integer part of } (n - \theta)/\lambda\} \quad (15)$$

Then the process evolves in the same way as the process with an infinite number of components when the total number of failures does not exceed  $N$ . If the total number of failures exceeds  $N$ , then it is assumed that all  $n$  components fail and the process ends. If the parameters are such that  $N < n - 1$ , this implies that it is impossible for  $N + 1, N + 2, \dots, n - 1$  components to fail. The saturation (15) is chosen so that the saturating model can be a good approximation to CASCADE and this is justified in subsections 6.1 and 6.2.

The standard result (10) above can be modified as follows to obtain the saturating model: The generating function  $G(t)$  for the total number of failures remains valid to order  $N$ . Write  $G^{[N]}(t)$  for the terms up to and including order  $N$  of  $G(t)$ . Then  $G^{[N]}(t)$  generates the probabilities of the total number of failures  $r$  for  $r \leq N$ . However, the sum of the probabilities generated by  $G^{[N]}(t)$  is  $G^{[N]}(1)$  and  $G^{[N]}(1) < 1$ . The probability generating function  $\hat{G}(t)$  for the saturating model can be obtained by making the probability of  $n$  failures equal to  $1 - G^{[N]}(1)$ :

$$\hat{G}(t) = G^{[N]}(t) + (1 - G^{[N]}(1))t^n \quad (16)$$

$$= \sum_{r=0}^N \theta(\theta + r\lambda)^{r-1} \frac{e^{-\theta-r\lambda}}{r!} t^r + (1 - G^{[N]}(1))t^n \quad (17)$$

The corresponding probability distribution is:

*Definition:*  $g(r, \theta, \lambda, n)$  is the probability that  $r$  components fail in the saturating generalized Poisson distribution model with initial disturbance mean failures  $\theta$ , cascading mean failures  $\lambda$ , and  $n$  components. For  $\theta < 0$ ,

$$g(r, \theta, \lambda, n) = 1; \quad r = 0 \quad (18)$$

$$g(r, \theta, \lambda, n) = 0; \quad r > 0 \quad (19)$$

For  $\theta \geq 0$ ,

$$g(r, \theta, \lambda, n) = \theta(r\lambda + \theta)^{r-1} \frac{e^{-r\lambda-\theta}}{r!} \quad ; \quad 0 \leq r \leq (n - \theta)/\lambda, r < n \quad (20)$$

$$g(r, \theta, \lambda, n) = 0; \quad (n - \theta)/\lambda < r < n, r \geq 0 \quad (21)$$

$$g(n, \theta, \lambda, n) = 1 - \sum_{s=0}^{n-1} g(s, \theta, \lambda, n) \quad (22)$$

The saturating form of the generalized Poisson distribution (20-22) limits the total number of failures to  $n$  even in the supercritical case and extends the range of parameters of the generalized Poisson distribution (10) to allow  $\lambda > 1$ .

There are other ways of normalizing or truncating the cascading process to avoid infinite quantities in the supercritical case. For example, one can normalize the number of failures  $M_k$  at stage  $k$  by their mean  $\lambda^k$  [23] or one can consider truncations motivated by not observing data in some

ranges [17, 14]. However, these methods are not suited to our application.

The mean number of failures in the saturating generalized Poisson distribution is

$$E[M] = \sum_{r=0}^N r\theta(\theta + r\lambda)^{r-1} \frac{e^{-\theta-r\lambda}}{r!} + n(1 - G^{[N]}(1)) \quad (23)$$

## 5. Review of CASCADE

This section summarizes the CASCADE model of probabilistic load-dependent cascading failure and the saturating quasibinomial distribution from [20].

The CASCADE model has  $n$  identical components with random initial loads. For each component the minimum initial load is  $L^{\min}$  and the maximum initial load is  $L^{\max}$ . For  $j=1,2,\dots,n$ , component  $j$  has initial load  $L_j$  that is a random variable uniformly distributed in  $[L^{\min}, L^{\max}]$ .  $L_1, L_2, \dots, L_n$  are independent.

Components fail when their load exceeds  $L^{\text{fail}}$ . When a component fails, a fixed amount of load  $P$  is transferred to each of the components.

To start the cascade, we assume an initial disturbance that loads each component by an additional amount  $D$ . Other components may then fail depending on their initial loads  $L_j$  and the failure of any of these components will distribute an additional load  $P \geq 0$  that can cause further failures in a cascade.

Now we define the normalized CASCADE model. The normalized initial load  $\ell_j$  is

$$\ell_j = \frac{L_j - L^{\min}}{L^{\max} - L^{\min}} \quad (24)$$

Then  $\ell_j$  is a random variable uniformly distributed on  $[0, 1]$ . Let

$$p = \frac{P}{L^{\max} - L^{\min}}, \quad d = \frac{D + L^{\max} - L^{\text{fail}}}{L^{\max} - L^{\min}} \quad (25)$$

Then the normalized load increment  $p$  is the amount of load increase on any component when one other component fails expressed as a fraction of the load range  $L^{\max} - L^{\min}$ . The normalized initial disturbance  $d$  is a shifted initial disturbance expressed as a fraction of the load range. Moreover, the failure load is  $\ell_j = 1$

The saturating quasibinomial distribution is given by:

*Definition:*  $f(r, d, p, n)$  is the probability that  $r$  components fail in the CASCADE model with normalized initial disturbance  $d$ , normalized load transfer amount  $p$ , and  $n$  components. For  $d < 0$ ,

$$f(r, d, p, n) = 1; \quad r = 0 \quad (26)$$

$$f(r, d, p, n) = 0; \quad r > 0 \quad (27)$$

For  $d \geq 0$ ,

$$f(r, d, p, n) = \binom{n}{r} d(rp + d)^{r-1} (1 - rp - d)^{n-r} \quad (28)$$

$$; \quad 0 \leq r \leq (1 - d)/p, \quad r < n \quad (28)$$

$$f(r, d, p, n) = 0; \quad (1 - d)/p < r < n, \quad r \geq 0 \quad (29)$$

$$f(n, d, p, n) = 1 - \sum_{s=0}^{n-1} f(s, d, p, n) \quad (30)$$

If  $np + d \leq 1$ , (28) and (30) reduce to the quasibinomial distribution introduced as an urn model by Consul [13]. Thus (28–30) extend the quasibinomial distribution to parameters with  $np + d > 1$ .  $np + d > 1$  corresponds to highly stressed systems with a significant probability of all components failing.

The distribution (26–30) can also be expressed using a saturation function  $\phi$  as follows [21]:

$$f(r, d, p, n) = \begin{cases} \binom{n}{r} \phi(d)(d + rp)^{r-1} (\phi(1 - d - rp))^{n-r}, & r = 0, 1, \dots, n-1 \\ 1 - \sum_{s=0}^{n-1} f(s, d, p, n), & r = n \end{cases} \quad (31)$$

where

$$\phi(x) = \begin{cases} 0 & ; x < 0 \\ x & ; 0 \leq x \leq 1 \\ 1 & ; x > 1 \end{cases} \quad (32)$$

Note that (31) uses  $0^0 \equiv 1$  and  $0/0 \equiv 1$  when needed.

## 6. Approximating CASCADE as a branching process

We first approximate the distribution of the total number of failures in CASCADE by the distribution of total number of failures in a saturating Poisson branching process. Then we show how the cascading failures in CASCADE can be approximated stage by stage by a Poisson branching process.

### 6.1. Approximating the distribution of the total number of failures

The total number of failures in the CASCADE model is distributed according to the saturating quasibinomial distribution (26)-(30). We prove that the saturating quasibinomial distribution can be approximated by the saturating generalized Poisson distribution (18)-(22).

Let  $n \rightarrow \infty$  and  $p \rightarrow 0$  and  $d \rightarrow 0$  in such a way that  $\lambda = np$  and  $\theta = nd$  are fixed. Then the appendix

[17] shows that the quasibinomial distribution tends to the generalized Poisson distribution. Hence for large  $n$  and for  $0 \leq r \leq (1-d)/p = (n-\theta)/\lambda$ , (28) may be approximated by (20).  $(1-d)/p = (n-\theta)/\lambda$  also implies that (29) may be replaced by (21). Then the preceding results imply that (30) tends to (22).

## 6.2. Branching process obtained from CASCADE

This subsection informally shows how failures in CASCADE arise in stages approximately as stages of a saturating branching process. The CASCADE model produces failures in stages  $i = 0, 1, 2, \dots$  where  $M_i$  is the number of failures in stage  $i$ . The following is a normalized version of the algorithm for CASCADE that can be derived from [20].

*Algorithm for normalized CASCADE model*

0. All  $n$  components are initially unfailed and have initial loads  $\ell_1, \ell_2, \dots, \ell_n$  determined as independent random variables uniformly distributed in  $[0, 1]$ .
1. Add the initial disturbance  $d$  to the load of component  $j$  for each  $j = 1, \dots, n$ . Initialize the stage counter  $i$  to zero.
2. Test each unfailed component for failure: For  $j = 1, \dots, n$ , if component  $j$  is unfailed and its load  $> 1$  then component  $j$  fails. Suppose that  $M_i$  components fail in this step.
3. If  $M_i = 0$ , stop; the cascading process ends.
4. If  $M_i > 0$ , then increment the component loads according to the number of failures  $M_i$ : Add  $M_i p$  to the load of component  $j$  for  $j = 1, \dots, n$ .
5. Increment the stage counter  $i$  and go to step 2

It is convenient throughout to restrict  $m_0, m_1, \dots$  to non-negative integers and to write

$$s_i = m_0 + m_1 + \dots + m_i \quad (33)$$

Consider the end of step 2 of stage  $i \geq 1$  in the CASCADE algorithm. The failures that have occurred are  $M_0 = m_0, M_1 = m_1, \dots, M_i = m_i$ , but the loads have not yet been incremented by  $m_i p$  in the following step 4. Let

$$\alpha_{i+1} = \phi\left(\frac{m_i p}{1-d-s_{i-1}p}\right) \quad (34)$$

where  $\phi$  is the saturation function defined in (32).

Suppose that  $d + s_{i-1}p \leq 1$ . Then the loads of the  $n - s_i$  unfailed components are uniformly distributed in  $[d + s_{i-1}p, 1]$ . This uniform distribution is conditioned on the  $n - s_i$  components not yet having failed. In the following step 4, the probability that the load increment of  $m_i p$

causes one of the unfailed components to fail is  $\alpha_{i+1}$  and the probability of  $m_{i+1}$  failures in the  $n - s_i$  unfailed components is

$$P[M_{i+1} = m_{i+1} | M_i = m_i, \dots, M_0 = m_0] = \binom{n-s_i}{m_{i+1}} \alpha_{i+1}^{m_{i+1}} (1-\alpha_{i+1})^{n-s_i-m_{i+1}}, \quad m_{i+1} = 0, 1, \dots, n-s_i \quad (35)$$

and the generating function for (35) is

$$(1 + \alpha_{i+1}(t-1))^{n-s_i} \quad (36)$$

Suppose that  $d + s_{i-1}p > 1$ . Then all the components must have failed on a previous step and  $P[M_{i+1} = m_{i+1} | M_i = m_i, \dots, M_0 = m_0] = 1$  for  $m_{i+1} = 0$  and vanishes otherwise. In this case  $\alpha_{i+1} = 0$  and (35) and (36) are again verified.

Let  $nd = \theta$  and  $np = \lambda$ . Then

$$\alpha_{i+1} = \phi\left(\frac{m_i \lambda}{n - \theta - s_{i-1} \lambda}\right) \quad (37)$$

There are three cases:

(1)  $s_{i-1} > (n - \theta)/\lambda$ . Then  $\alpha_{i+1} = 0$ , (36) evaluates to 1 and  $P[M_{i+1} = 0 | M_i = m_i, \dots, M_0 = m_0] = 1$ . Case 1 is an already saturated case corresponding to all components failing in stage  $i - 1$  or previous stages.

(2)  $s_{i-1} \leq (n - \theta)/\lambda$  and  $s_i = m_i + s_{i-1} \geq (n - \theta)/\lambda$ . Then  $\alpha_{i+1} = 1$ , (36) evaluates to  $t^{n-s_i}$  and  $P[M_{i+1} = n - s_i | M_i = m_i, \dots, M_0 = m_0] = 1$ . Case 2 is a saturating case corresponding to all components failing in stage  $i$ .

(3)  $s_i = m_i + s_{i-1} < (n - \theta)/\lambda$ . Then

$$\alpha_{i+1} = \frac{m_i \lambda}{n - \theta - s_{i-1} \lambda}$$

Let  $n \rightarrow \infty$  and  $p \rightarrow 0$  so that  $np = \lambda$ . Since

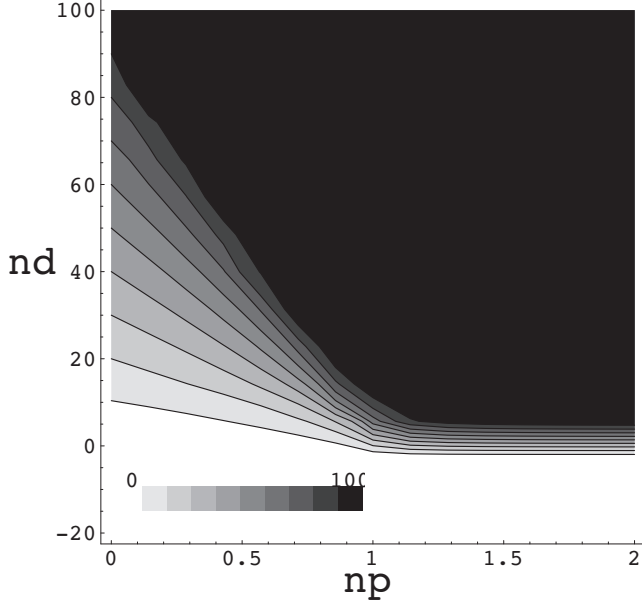
$$(1 + \alpha_{i+1}(t-1))^{n-s_i} \rightarrow e^{m_i \lambda (t-1)} \text{ as } n \rightarrow \infty \quad (38)$$

we approximate (36) by

$$\left(e^{m_i \lambda (t-1)}\right)^{[n-s_i-1]} + t^{n-s_i} \left(1 - \left(e^{m_i \lambda (t-1)}\right)^{[n-s_i-1]}\right) (1) \quad (39)$$

That is, the approximation is

$$P[M_{i+1} = m_{i+1} | M_i = m_i, \dots, M_0 = m_0] = \begin{cases} \frac{(m_i \lambda)^{m_{i+1}}}{m_{i+1}!} e^{-m_i \lambda}, & m_{i+1} = 0, 1, \dots, n - s_i - 1 \\ 1 - \sum_{k=0}^{n-s_i-1} \frac{(m_i \lambda)^k}{k!} e^{-m_i \lambda}, & m_{i+1} = n - s_i. \end{cases} \quad (40)$$



**Figure 1. Average number  $\langle r \rangle$  of components failed in CASCADE as a function of  $np$  and  $nd$  for  $n = 100$ . Lines are contours of constant  $\langle r \rangle$ . White indicates  $< 10$  failures and black indicates  $> 90$  failures.**

According to (38), for fixed  $r$ , the approximation (39) becomes exact as  $n \rightarrow \infty$ . That is, the coefficient of  $t^r$  in (39) tends to the coefficient of  $t^r$  in (36) as  $n \rightarrow \infty$ . However, the approximation (39) is inaccurate for the coefficient of  $t^r$  when  $r = n - s_i$  or  $r$  is close to  $n - s_i$ .

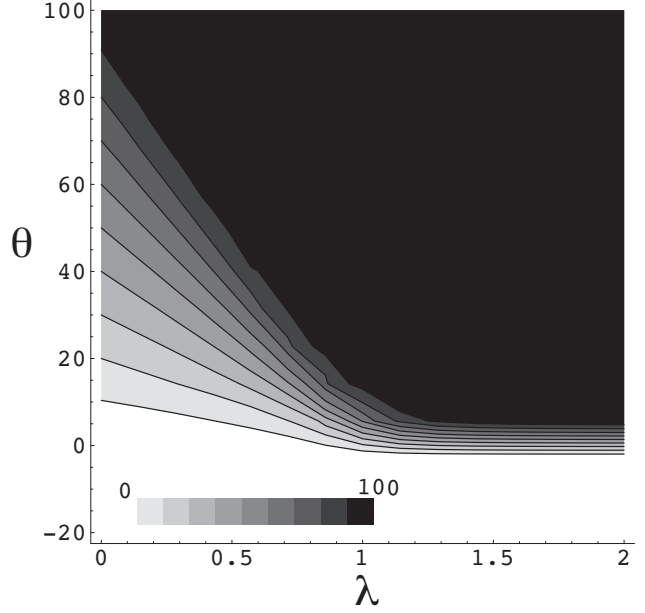
Since  $e^{m_i \lambda (s-1)} = (e^{\lambda (s-1)})^{m_i}$ , (39) or (40) is the distribution of the sum of  $m_i$  independent Poisson random variables with rate  $\lambda$  with saturation occurring when the total number of failures exceeds  $n$ . Thus we can consider each failure as independently causing other failures in the next stage according to a saturating Poisson process.

A similar approximation applies at stage zero. Suppose that in step 2 of stage zero in the CASCADE algorithm there are  $m_0$  failures due to the initial disturbance  $d$ . The probability that the load increment of  $d$  causes one of the components to fail is  $\phi(d)$  and the probability of  $m_0$  failures in the  $n$  components is given by:

$$\binom{n}{m_0} \phi(d)^{m_0} (1 - \phi(d))^{n-m_0} \quad (41)$$

Let  $n \rightarrow \infty$  and  $d \rightarrow 0$  so that  $nd \rightarrow \theta$ . Then we approximate (41) by the saturating Poisson distribution

$$P[M_0 = m_0] = \begin{cases} \frac{\theta^{m_0}}{m_0!} e^{-\theta} & , m_0 = 0, 1, \dots, n-1 \\ 1 - \sum_{k=0}^{n-1} \frac{\theta^k}{k!} e^{-\theta} & , m_0 = n. \end{cases} \quad (42)$$



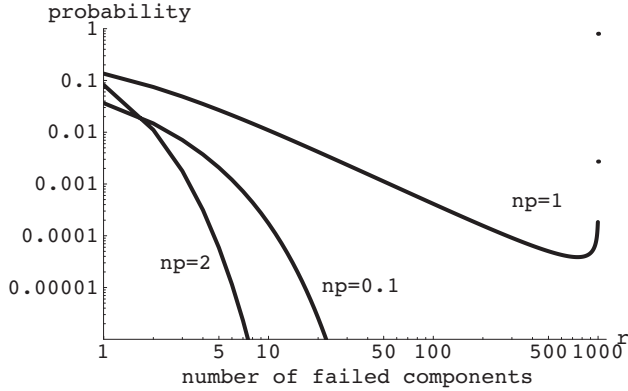
**Figure 2. Average number  $\langle r \rangle$  of components failed in saturating generalized Poisson distribution as a function of  $\lambda$  and  $\theta$  for  $n = 100$ . Lines are contours of constant  $\langle r \rangle$ . White indicates  $< 10$  failures and black indicates  $> 90$  failures.**

The approximations (40) and (42) show that the number of failures in each stage are, for large  $n$  and small  $p$  and  $d$ , governed by a saturating Poisson branching process with mean  $\lambda = np$ , except that on the first step the mean is  $\theta = nd$ . The approximation does not necessarily imply that concepts natural to the branching process translate directly to the CASCADE model. For example, each failure in CASCADE may be attributed to load increases caused by many previous failures, whereas it is natural to attribute each failure in a branching process to a single previous failure.

The mean number of failures in the CASCADE and the saturating generalized Poisson distribution as a function of  $\theta$  and  $\lambda$  are compared in Figures 1 and 2. Scans corresponding to load increase with  $d = p$  and  $\theta = \lambda$  are compared in Figures 3 and 4. Note the closeness of the approximation for small and moderate  $r$  and the expected inaccuracy of the approximation near  $r = n$ .

## 7. Discussion

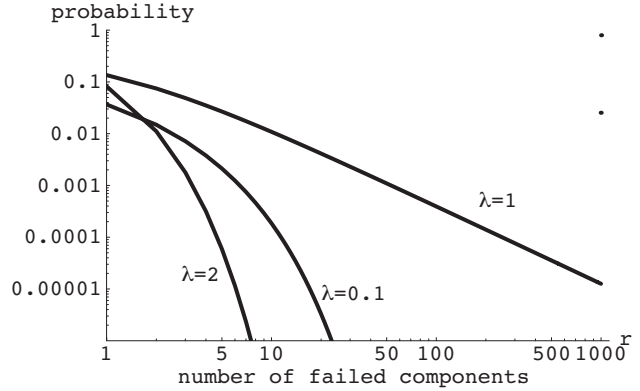
Large power system blackouts typically involve a cascading series of failures or outages in which the system becomes weaker or more stressed as the cascade proceeds. There are many ways in which failure or outage of a compo-



**Figure 3. CASCADE probability distributions of total number of failures on log-log plot.  $n = 1000$ . Note that the probability of 1000 components failing is 0.003 for  $np = 1$ , and 0.798 for  $np = 2$ .**

nent can adversely affect other components and make their failure more likely. For example, outage of a line can make more likely the failure of other components via redistribution of load, relay or control system misoperation [28], transient phenomena, or operator or planning error. Moreover, all these interactions generally become stronger as power system loading is increased and the significant interactions become more numerous. High loading tends to make interactions more nonlinear, harder to conceive of in advance and much more likely to cause further failures since margins are smaller. In the terminology of Perrow [27], highly loaded power systems are more complex and tightly coupled. The diversity of components and interactions in the power system is highly simplified in the CASCADE model to uniform components that interact in a uniform and simple way with all the other system components. The branching process model is even further abstracted in that component failures cause other failures by an unspecified mechanism. While this paper does claim to capture salient features of cascading blackouts in both of these simple models, it should be acknowledged that substantial work is needed to determine the detailed similarities and differences between these models and real blackouts via statistical measurements and simulations. Estimating  $\lambda$  from a simulation of cascading outages is considered in [8]. The consequences of nonuniform interactions between components or interactions limited to a subset of other components also needs to be examined in future work.

The CASCADE model captures the weakening of system as the cascade proceeds and reproduces some qualitative features of blackout size probability distributions observed in blackout data and simulations [19, 9, 7]. Since this paper shows that CASCADE is well approximated by



**Figure 4. Saturating generalized Poisson probability distributions of total number of failures on log-log plot.  $n = 1000$ . Note that the probability of 1000 components failing is 0.025 for  $\lambda = 1$ , and 0.797 for  $\lambda = 2$ .**

a branching process and the saturating generalized Poisson distribution, the saturating generalized Poisson distribution also reproduces the same qualitative features of blackout size probabilities.

The approximation of CASCADE by the branching process allows the parameters of the two models to be related. Thus

$$\lambda = np \quad (43)$$

$$= \frac{nP}{L_{\max} - L_{\min}} \quad (44)$$

Recall that in CASCADE,  $p$  is the normalized load transfer amount and  $n$  is the number of components. (43) can be used to reinterpret  $p = \lambda/n$  in the branching process as the probability that a component failure causes the failure of a specific other component. This is an important interpretation in contexts in which there is a cascading dependency between components that is not naturally expressed as an increment in loading.

The criterion (14) for minimizing cascading failure can be reexpressed using (43) as  $np < \lambda_{\max}$ . Then even if  $p$  is very small, large  $n$  can cause cascading failure. This suggests that numerous rare interactions can be equally influential in causing cascading failure as a smaller number of likely interactions. More generally, one can speculate that a design change that introduced a large number of unlikely failure interactions (plausibly similar to large  $n$ ) could make cascading failure more likely, despite high reliability (low  $p$ ). It is conceivable that coupling infrastructures together such as controlling the power system over an internet or certain types of global control schemes could make the system more vulnerable to cascading failure in this fashion. It is also interesting to note that many traditional power system controls are designed to reduce interactions by deliberate



separation in distance, frequency, and time scale.

The criterion (14) for minimizing cascading failure can be reexpressed using (44) as

$$\lambda = \frac{nP}{L_{\max} - L_{\min}} < \lambda_{\max} \quad (45)$$

There are several ways to represent system load increase in CASCADE [20]. One of these ways increases average component load by increasing  $L_{\min}$ . Then (45) shows how this form of load increase affects the criterion limiting the risk of cascading failure. The relation (45) between  $\lambda$  and  $L_{\min}$  is nonlinear.

## 8. Conclusion

We introduce a saturating form of the generalized Poisson distribution and show that it approximates the distribution of total number of failures in the CASCADE model of load-dependent cascading failure. Moreover, successive failures in stages of CASCADE can be approximated by corresponding stages of a saturating Poisson branching process. The approximation of CASCADE as a branching process yields insights into the power tails and criticality observed in CASCADE. The branching process approximation is simpler and more analytically tractable than CASCADE while retaining qualitative features of load-dependent cascading failure. Moreover, at criticality the universality of the  $-\frac{3}{2}$  power law in the probability distribution of the total number of failures in a branching process suggests that this is a signature for this type of cascading failure. The  $-\frac{3}{2}$  power law is approximately consistent with North American blackout data and blackout simulation results.

Criticality or supercriticality in the branching process implies a high risk of catastrophic and widespread cascading failures. Maintaining sufficient subcriticality in the branching process according to a simple criterion (14) would limit the propagation of failures and reduce this risk. The approximation of CASCADE as a branching process allows the criterion to be expressed in terms of system loading (45). However, implementing the criterion to reduce the risk of catastrophic cascading failure would require limiting the system throughput and this is costly. Managing the tradeoff between the certain cost of limiting throughput and the rare but very costly widespread catastrophic cascading failure may be difficult. Indeed [18, 4, 5] maintain that for large blackouts, economic, engineering and societal forces may self-organize the system to criticality and that efforts to mitigate the risk should take account of these broader dynamics [6].

Our emphasis on limiting the propagation of system failures after they are initiated is complementary to more standard methods of mitigating the risk of cascading failure by

reducing the risk of the first few likely failures caused by an initial disturbance as for example in [10].

The branching process approximation does capture some salient features of loading dependent cascading failure and suggests an approach to reducing the risk of large cascading failures by limiting the average propagation of failures. However, much work remains to establish the correspondence between these simplified global models and the complexities of cascading failure in real systems.

## 9. Acknowledgements

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## A. Approximating quasibinomial distribution

The generalized Poisson distribution is [17, 16]

$$G(r, \theta, \lambda) = \theta(r\lambda + \theta)^{r-1} \frac{e^{-r\lambda - \theta}}{r!} \quad (46)$$

for  $\lambda \leq 1$  and  $\theta > 0$ . We use Consul's derivation [16] that the quasibinomial distribution tends to the generalized Poisson distribution. The quasibinomial distribution is

$$\binom{n}{r} d(rp + d)^{r-1} (1 - rp - d)^{n-r} \quad (47)$$

for  $d + np \leq 1$  and  $0 < d < 1$ .

If  $d \rightarrow 0$ ,  $p \rightarrow 0$  and  $n$  increases without limit such that  $nd = \theta$  and  $np = \lambda$ , then (47) can be written in the form

$$\frac{nd(rnp + nd)^{r-1}}{r!} \frac{n!}{(n-r)! n^r} \left[ 1 - \frac{r\lambda + \theta}{n} \right]^{n-r} \quad (48)$$

which can be rewritten as

$$\begin{aligned} & \theta(r\lambda + \theta)^{r-1} \frac{e^{-r\lambda - \theta}}{r!} \left[ 1 - \frac{r\lambda + \theta}{n} \right]^n e^{r\lambda + \theta} \\ & \left[ \left( 1 - \frac{1}{n} \right) \left( 1 - \frac{2}{n} \right) \dots \left( 1 - \frac{r-1}{n} \right) \right] \left[ 1 - \frac{r\lambda + \theta}{n} \right]^{-r} \\ & = G(r, \theta, \lambda) \left[ 1 - \frac{(r\lambda + \theta)^2}{2n} \right] \\ & \quad \left[ 1 + \frac{2r(r\lambda + \theta) - r(r-1)}{2n} \right] + O(n^{-2}) \\ & = G(r, \theta, \lambda) \left[ 1 + \frac{2r(r\lambda + \theta) - r(r-1) - (r\lambda + \theta)^2}{2n} \right] \\ & \quad + O(n^{-2}) \\ & = G(r, \theta, \lambda) \left[ 1 + \frac{r - (r\lambda - 1) + \theta^2}{2n} \right] + O(n^{-2}) \quad (49) \end{aligned}$$

Hence the generalized Poisson distribution is the limit of the quasibinomial distribution.

Examination of the  $1 + O(n^{-1})$  factor in (49) suggests that the approximation improves for  $\lambda \approx 1$  and only slowly gets worse for larger  $r$ . For  $\lambda \not\approx 1$ , the  $1 + O(n^{-1})$  factor suggests that the approximation gets worse for larger  $r$ .