Observations on the Geometry of Saddle Node Bifurcation and Voltage Collapse in Electrical Power Systems

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Abstract—**Saddle node bifurcation is a generic instability of parameterized differential equation models. We describe the bifurcation geometry and some implications for the study of voltage collapse in electric power systems. The initial direction in state space of dynamic voltage collapse can be calculated from a right eigenvector of a static power system model. The normal vector to the bifurcation set in parameter space is a simple function of a left eigenvector and is expected to be useful in emergency control near bifurcation and computing the minimum distance to bifurcation in parameter space.**

I. Introduction

The purpose of this paper is to make some observations about the geometry of generic saddle node bifurcations and explain how the observations are useful in modeling and calculations for voltage collapse in electric power systems. Voltage collapse is an instability of heavily loaded electric power systems that leads to declining voltages and blackout. It is associated with bifurcation and reactive power limitations of the power system. Most of the current approaches to modeling and predicting voltage collapse are represented in [1] and [2].

Power system generators reaching reactive power limits and discrete contingencies, such as loss of a power line, are thought to be important in voltage collapse but they are hard to study in general power system models because they cause changes in the form of the equations governing the system. In this paper we consider the simpler problem of voltage collapse when generator limitations do not change and system parameters vary slowly (rather than discretely) to cause saddle node bifurcation and the subsequent voltage collapse. (Note that these assumptions may apply *after* a discrete contingency.) A natural and general power system model under these assumptions is a set of differential equations with slowly varying parameters, and very general conclusions [3] about the structure of voltage collapse may be made by exploiting generic bifurcation theory [4].

One of the difficulties in applying bifurcation theory to power systems is that the power system equations have multidimensional state and parameter vectors so that the bifurcation geometry is multidimensional. A description of this multidimensional geometry for the saddle node bifurcation follows readily from known results in generic bifurcation theory [4]–[7] and has useful consequences for modeling voltage collapse and calculating the proximity to voltage collapse. One of the more striking consequences is that the direction of the initial dynamic collapse may be studied using a static model of the power system. Much of the paper describes the bifurcation geometry in state space and particularly the engineering implications of right and left eigenvectors associated with the bifurcation. Also useful is the geometry in parameter space of the bifurcation set, which is the set of critical parameters corresponding to bifurcations. We derive a simple formula depending on a left eigenvector for the normal vector to the bifurcation set and suggest how this information could be useful in monitoring and avoiding voltage collapse.

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II. Voltage Collapse and Generic Saddle Node Bifurcation

We model the power system as differential equations with a vector λ of slowly varying parameters:

$$\dot{x} = f(x, \lambda), \ x \in \boldsymbol{R}^n, \ \lambda \in \boldsymbol{R}^m. \tag{1}$$

x is a state vector that includes bus voltage magnitudes and angles. λ is typically a vector of real and reactive load powers. The load powers vary with time so that λ is a function of time t and (1) may be regarded as differential equations parameterized by the single parameter t:

$$\dot{x} = f(x, \lambda(t)), \ x \in \boldsymbol{R}^n, \ t \in \boldsymbol{R}. \tag{2}$$

Voltage collapse is associated with saddle node bifurcation of (1) or (2). We now sketch results from [3] based on Sotomayor's generic bifurcation theory [4], which give simple models for voltage collapse before and after the saddle node bifurcation. Dynamical systems terminology in the sequel is explained in [3] and [8]. We assume that (2) is in Sotomayor's class of generic one-parameter systems and exploit his results about the structure of saddle node bifurcations in these systems. The genericity implies that the saddle node bifurcations are robust (also see [9]) and expected to occur in practice. We use the saddle node bifurcation to model voltage collapse because the only other bifurcation generic in one-parameter families of systems is the Hopf bifurcation, which leads to an oscillatory instability as opposed to the monotonic decrease observed in voltage collapse.

Before bifurcation, the system has a stable equilibrium x_0 and all the eigenvalues of the Jacobian $D_x f(x_0, \lambda)$ have negative real parts. As the parameter λ slowly varies, the stable equilibrium x_0 varies and the system state x tracks x_0 so that x_0 is also the system operating point. Thus a static (or quasistatic) model $0 = f(x, \lambda)$ is used before the bifurcation. The saddle node bifurcation consists of the stable equilibrium x_0 coalescing with a nearby unstable type one equilibrium x_1 and disappearing, causing the system to lose stability. We write $\lambda_* = \lambda(t_*)$ for the critical value of the parameter vector at bifurcation and x_* for the corresponding equilibrium formed by x_0 and x_1 coalescing. The Jacobian $D_x f(x_*, \lambda_*)$ is singular and has a unique simple zero eigenvalue with a corresponding right eigenvector v_* so that $D_x f(x_*, \lambda_*) v_* = 0$. At bifurcation, x_* is unstable and the system dynamics may be approximated by the system state moving along the particular trajectory W_+^c, which is the unstable part of the center manifold of x_*. If W_+^c points in a direction in state space so that voltage magnitudes decrease as the system state moves along W_+^c, then we identify voltage collapse with the movement along W_+^c. This is the center manifold model for the dynamics of voltage collapse [3]. See [3] for a detailed explanation and an example.

III. Right Eigenvector v_*

The right eigenvector v_* is tangent to W_+^c at x_* so that v_* defines the direction in state space of the initial dynamics of voltage collapse. Any of the state variables can "collapse" as the system dynamics move in the direction v_*, and the extent to which a variable collapses is given by the relative magnitude of the corresponding component of v_* [3]. The buses at which voltage magnitudes will fall most quickly have the largest components of v_*. Indeed, the components of v_* that are negative and of sufficient absolute magnitude can be used to identify a group of buses at which the collapse is most severe [10], [11].

As well as defining the direction of the initial dynamics at bifurcation, v_* has a useful interpretation in the static model $0 = f(x, \lambda)$. As the bifurcation occurs, the equilibria x_0 and x_1 coalesce and v_* is the asymptotic direction in which x_0 and x_1 approach one another. For a proof, study the Liapunov–Schmidt reduction [5], [6], [12], which completely solves the local geometry of $0 = f(x, \lambda)$ near (x_*, λ_*).

When a saddle node bifurcation is close (λ near λ_*), x_1 is near x_0, and $D_x f(x_0, \lambda)$ has a unique, simple negative eigenvalue μ of smallest absolute magnitude with a corresponding right eigenvector v. $\mu \to 0$ and $v \to v_*$ as $\lambda \to \lambda_*$. Since v is a continuous function of λ for λ near λ_*, v lies approximately along the line joining x_0 and x_1 when the system is close to bifurcation. Thus given x_0, the best estimate for the direction in which to find x_1 is given by v or $-v$. This is useful because one can choose initial conditions for the numerical calculation of x_1 along the line through x_0 in the direction v.

IV. Studying Initial Voltage Collapse Dynamics with a Static Model

Consider the relationship between the static model (3) and the dynamic model (4):

$$0 = g(x, \lambda) \tag{3}$$

$$\dot{x} = f(x, \lambda) = h(g(x, \lambda)) \tag{4}$$

where $h(0) = 0$. Then solutions of (3) are equilibria of (4), and bifurcation of solutions of (3) at (x_*, λ_*) implies bifurcation of equilibria of (4) at (x_*, λ_*). Moreover, they have the *same* right eigenvector v_* because $D_x g v_* = 0$ implies that $D_x f v_* = DhD_x g v_* = 0$. (If the Jacobian Dh is globally invertible and $h(x) = 0$ iff $x = 0$, then *all* saddle-node bifurcations of (4) are also saddle-node bifurcations of (3).)

Thus studying bifurcations of (3) also studies bifurcations of a whole class of dynamic models (4) whose steady-state behavior is (3). If we assume the center manifold model for collapse at bifurcation, then v_* gives the initial direction of the dynamic collapse. Thus we can calculate v_* and the initial collapse direction from the static model (3) when we are given only the general form, and not the particular details, of (4). The rate of the collapse (slow then fast) is known qualitatively from [3].

Now we show how these ideas apply to a basic power system model. Let y be a vector of load bus voltage angles and magnitudes and let δ_G be a vector of generator voltage angles. Then a static model (load flow equations) is

$$\begin{aligned} 0 &= g_1(\delta_G, y) \\ 0 &= g_2(\delta_G, y, \lambda) \end{aligned} \tag{5}$$

where g_1 describes real power balance at the generators and g_2 describes real and reactive power balance at the loads. The parameter vector λ represents change in load power demands.

A dynamic model that extends (5) by including generator swing dynamics and load dynamics is

$$\begin{aligned} \dot{\delta}_G &= \omega \\ \dot{\omega} &= g_1(\delta_G, y) - \Delta\omega \\ \dot{y} &= h_2(g_2(\delta_G, y, \lambda), \omega). \end{aligned} \tag{6}$$

Here, h_2 defines any dynamic load model that depends on the real and reactive power balance at each load and frequency ω (see [3] and [13] for examples). Little is known about such load dynamics, and a convincing function h_2 is hard to obtain. (We do not consider the models of the more general form $\dot{y} = h_2(g_2(\delta_G, y, \lambda), \omega, y)$.)

The Jacobians of (5) and (6) are

$$\begin{pmatrix} D_{\delta_G} g_1 & D_y g_1 \\ D_{\delta_G} g_2 & D_y g_2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & I & 0 \\ D_{\delta_G} g_1 & -\Delta & D_y g_1 \\ D_x h_2 D_{\delta_G} g_2 & D_\omega h_2 & D_x h_2 D_y g_2 \end{pmatrix}.$$

The form of the Jacobians shows that when the static model (5) bifurcates at $(\delta_{G_*}, y_*, \lambda_*)$ with $v_*^{\text{stat}} = (\delta_G^0, y^0)$, the dynamic model (6) bifurcates at $(\delta_{G_*}, 0, y_*, \lambda_*)$ with $v_*^{\text{dyn}} = (\delta_G^0, 0, y^0)$. Since v_*^{dyn} defines the initial collapse direction and v_*^{dyn} is immediately obtainable from v_*^{stat}, the initial collapse direction may be immediately deduced from the static model (5). The useful point is that the initial collapse direction does not depend on the details of the load dynamics in h_2 and we can use the simpler static model (5) to study the bifurcation of the dynamic model (6) and its initial collapse direction.

V. Left Eigenvector w_* in State Space

At the bifurcation, the Jacobian $D_x f(x_*, \lambda_*)$ of (1) has a unique simple zero eigenvalue with corresponding left eigenvector w_* so that $w_* D_x f(x_*, \lambda_*) = 0$. (Note that w_* is a row vector.) $D_x f(x_*, \lambda_*)$ has $n - 1$ eigenvalues with negative real parts and the right generalized eigenvectors corresponding to these eigenvalues span a hyperplane $TW^s(x_*)$ through x_* in the state space $\boldsymbol{R}^n$. w_* may be interpreted geometrically as the normal vector to $TW^s(x_*)$. (To demonstrate this we need to show that $w_* v = 0$ for each of the right generalized eigenvectors v with nonzero eigenvalue. If the eigenvalue corresponding to the eigenvector v is $\eta \neq 0$, then this follows from $0 = w_* D_x f(x_*, \lambda_*) v = \eta w_* v$. If v is a generalized eigenvector of order k, use the more elaborate formula $0 = w_*(D_x f(x_*, \lambda_*) - \eta I)^k v = (-1)^k \eta^k w_* v$.) It follows that w_* is the normal vector to the stable manifold $W^s(x_*)$, which is an invariant hypersurface passing through x_* whose tangent hyperplane at x_* is $TW^s(x_*)$.

Near bifurcation, the Jacobian $D_x f(x_0, \lambda)$ has a negative eigenvalue μ of smallest absolute magnitude. Section III explained how the right eigenvector v corresponding to μ approximates the direction from x_0 to x_1. Now we similarly interpret the left eigenvector w of $D_x f(x_0, \lambda)$ corresponding to μ. The part of the stability boundary (basin boundary) closest to x_0 is $W^s(x_1)$, the stable manifold of x_1. $W^s(x_1)$ is a hypersurface since x_1 is type one and $D_x f(x_1, \lambda)$ has exactly one positive eigenvalue μ_1. The normal vector to $W^s(x_1)$ at x_1, and to its tangent plane $TW^s(x_1)$ at x_1, is the left eigenvector w_1 corresponding to μ_1. As $\lambda \to \lambda_*$, μ and $\mu_1 \to 0$ and w and $w_1 \to w_*$. Thus near bifurcation w approximates w_1. If we approximate $W^s(x_1)$ near x_1 by $TW^s(x_1)$, then the direction of the shortest line segment from x_0 to $W^s(x_1)$ is approximated by w_1. Hence w approximates the direction of the shortest line segment joining x_0 to the stability boundary. This direction is that of the smallest perturbation in the state vector x from x_0 that can destabilize the system. The approximation of this direction could be improved by calculating x_1, $D_x f(x_1, \lambda)$, and w_1.

VI. The Normal Vector to the Bifurcation Set in Parameter Space

The saddle node bifurcation set Σ in parameter space $\boldsymbol{R}^m$ is the set of λ that yields a saddle node bifurcation of (1). If $\lambda_* \in \Sigma$ yields one of the generically occurring saddle nodes, then Lemma 1 below shows that Σ is a smooth hypersurface near λ_* and that the normal vector to Σ at λ_* is $w_* D_\lambda f(x_*, \lambda_*)$. The calculation of $D_\lambda f(x_*, \lambda_*)$ is particularly straightforward for power system models parameterized by load powers because the load powers λ appear linearly so that $f(x, \lambda) = g(x) + L\lambda$, where L is an $n \times m$ constant matrix. Then $D_\lambda f = L$ and the normal vector to Σ is simply $w_* L$. The normal vector to Σ is useful for monitoring and controlling the power system because it defines the most critical direction in parameter space for causing or avoiding the bifurcation. Before presenting Lemma 1, we outline two applications.

1) (Emergency load shedding at or near bifurcation.) At bifurcation $w_* L$ defines the normal vector to Σ and the direction in parameter space in which it is most effective to move in order to avoid the bifurcation. Thus the loads that it is most effective to shed correspond to the larger entries in the vector $w_* L$. Shedding loads corresponding to the smaller entries tends to move λ in a direction more parallel to Σ rather than away from Σ. See [14] for an example and the generalization to avoiding the bifurcation by varying parameters other than load powers.

2) (Estimation of minimum distance to bifurcation in parameter space.) Both the continuation and direct calculation methods of finding the bifurcation point (x_*, λ_*) require some assumption on how the load parameters λ will vary with time (e.g., [11], [15]–[17]). These methods work by assuming a particular curve $\lambda(t)$ in parameter space and then calculating where the curve intersects Σ and the bifurcation occurs. It is often sensible to assume a particular pattern of load increase to define the curve and measure the distance to bifurcation along the curve. However, it is also useful to estimate the line segment $\lambda_0 \lambda_*$ in parameter space with the minimum distance to Σ [18]–[23]. This line segment has endpoints λ_0 (current operating λ) and λ_* (nearest point on Σ to λ_0). The direction of $\lambda_0 \lambda_*$ is the worst case parameter variation for causing bifurcation and $|\lambda_* - \lambda_0|$ is the minimum distance to bifurcation. The key point is that $\lambda_0 \lambda_*$ is normal to Σ so that $w_* L$ at λ_* is parallel to $\lambda_0 \lambda_*$; algorithms that estimate $|\lambda_* - \lambda_0|$ will explicitly or implicitly use this parallelism [21]–[23]. Initial estimates of λ_* and $|\lambda_* - \lambda_0|$ for these algorithms can be obtained as follows. We calculate the left eigenvector w of $D_x f(x_0, \lambda_0)$. If the system is close to bifurcation, then w approximates w_* and wL approximates the direction of $\lambda_0 \lambda_*$ so that an estimate λ'_* of λ_* may be calculated by increasing λ from λ_0 in the direction wL until bifurcation at λ'_*. The approximate minimum distance to bifurcation is then $|\lambda'_* - \lambda_0|$. We also note the possibility of improving this estimate as follows. The left eigenvector w'_* of $D_x f(x'_*, \lambda'_*)$ can be computed from $D_x f(x'_*, \lambda'_*)$ or obtained as a by-product of the calculation of λ'_* by the methods of [11] and [16]. A new estimate λ''_* of λ_* may be calculated by increasing λ from λ_0 in the direction $w'_* L$ until bifurcation at λ''_*, yielding a new approximate minimum distance $|\lambda''_* - \lambda_0|$. Further iterations of this procedure yield a method of computing λ_* and $|\lambda_* - \lambda_0|$ as described and exemplified in [22]. See [21] and [23] for other algorithms, theory, and examples.

Lemma 1: Suppose (1) has a saddle node bifurcation at (x_*, λ_*) with $D_x f(x_*, \lambda_*)$ having a unique simple zero eigenvalue with corresponding right and left eigenvectors v_*, w_* and satisfying the transversality conditions $w_* D_\lambda f(x_*, \lambda_*) \neq 0$ and $w_* D_{xx} f(x_*, \lambda_*)(v_*, v_*) \neq 0$. Write Σ for the bifurcation set of (1). Then there is an open set $U \ni \lambda_*$ such that $S = \Sigma \cap U$ is a smooth hypersurface and the normal vector to S at λ_* is $w_* D_\lambda f(x_*, \lambda_*)$.

Proof: The assumptions imply (see [4, 3.1] or [5, 6.2]) that there is an open set $U \ni \lambda_*$ such that S is a smooth hypersurface and that the bifurcating equilibria near (x_*, λ_*) are given by $(u(\lambda'_*), \lambda'_*)$ where u is a smooth function $u: S \to R^n$. For any $\lambda'_* \in S$ we have $f(x'_*, \lambda'_*) = 0$ where $x'_* = u(\lambda'_*)$. Let $d\lambda'_*$ be an arbitrary infinitesimal displacement of λ'_* in S (i.e., any one-form in $T^* S_\lambda$). Then $D_x f dx'_* + D_\lambda f d\lambda'_* = 0$ and $w_* D_\lambda f d\lambda'_* = -w_* D_x f dx'_* = 0$. Since $d\lambda'_*$ is arbitrary, $w_* D_\lambda f$ is the normal vector to S at λ_*. □

Lemma 1 generalizes a result for symmetric Jacobians in [7] and can also be deduced from [5, 6.2, theorem 2.1]. Note that the transversality conditions assumed in Lemma 1 are satisfied by the saddle node bifurcations occurring in Sotomayor's generic class of one-parameter systems [4] and hence the saddle node bifurcations considered in this paper. Lemma 1 gives a geometric interpretation in parameter space of the quantity appearing in the first of the two transversality conditions. Kwatny *et al.* [6] propose determining critical collapse parameters by calculating $\gamma: U \to R$ so that $\gamma^{-1}(0) = S$. ($\gamma(\lambda)$ is the first term of the Taylor expansion of $\phi(a, \lambda)$ with respect to a where $\phi(a, \lambda)$ is the Lyapunov–Schmidt reduction of $f(x, \lambda) = 0$.) Kwatny's method may be related to Lemma 1 by noting that $w_* D_\lambda f(x_*, \lambda_*) = D_\lambda \phi(a_*, \lambda_*) = D_\lambda \gamma(\lambda_*)$.

VII. Conclusions

We describe the multidimensional geometry of generically occuring saddle node bifurcations and some implications for the study of voltage collapse in electric power systems. Our observations are general and should be useful in studying generic instabilities of other systems modeled by parameterized differential equations.

At the bifurcation a particular right eigenvector v_* is both the initial direction of the dynamic collapse and the asymptotic direction in which the closest unstable equilibrium x_1 approaches the stable operating point x_0. If a general form for the load dynamics is assumed, then v_* is calculable from static power system models and these models are sufficient to estimate the initial direction of the dynamic collapse. In particular, this direction is independent of the details of the load dynamics. Near the bifurcation, particular right and left eigenvectors v and w can be calculated from the Jacobian at x_0. v approximates the direction from x_0 to x_1 and w approximates the closest direction to the stability boundary.

In parameter space, $w_* D_\lambda f$ is the normal vector to the bifurcation set. w_* is a particular left eigenvector of the Jacobian at bifurcation and $D_\lambda f$ is a constant matrix for typical power system models parameterized by load powers. The normal vector to the bifurcation set is expected to be useful in emergency control near bifurcation and computing the minimum distance to bifurcation in parameter space.

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Time-Domain Analysis of Switched Networks Containing Periodically Operated Switches

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Abstract—**A simple approach for the analysis of linear systems containing periodically operated switches is described. It is based on computing the system's impulse response. Then, this impulse response series is reduced into a rational function form that makes the evaluation of its frequency spectrum quite easy. Illustrative examples are given and show that, as long as the incoming bandwidth is less than $f_s/2$ (where $f_s = 1/T$, T is the period of the periodically switched network), then the impulse response approach yields exactly the system's frequency response.**

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I. Introduction

Several methods have been described for the analysis of linear circuits containing periodically operated switches [1], [2]. However, the approach used in these methods depends heavily on matrix manipulations as they require matrix inversion as well as exponentiation, for every computed frequency point. In [3], an alternate approach based upon the computation of the conversion function has been described for networks containing ideal switches with zero switching instant. However, in the case of networks employing lossy switches, as well as in the case of multistage coupling networks separated by switches with different clock rates, no solution is known. Quite recently [4], a novel time-domain based approach has been described for the analysis and design of multirate digital systems. It has been concluded that this approach is far more efficient than the standard frequency-domain analysis. In this paper, we extend this approach to the analysis and design of linear networks containing periodically operated switches. The main feature of this new approach is that it avoids involved matrix operations, especially matrix inversion at every frequency point. The only limitation of this method is that it yields almost the exact response up to half the sampling frequency.

II. Exact Analysis of Networks with Periodically Operated Switches

Consider a periodically switched linear network, in which the switches change states with a period T s. In [1], an exact analysis is given for the case where the period T is divided into only two phases. For the general case where T is broken into N phases, a similar approach can be followed. That is, during the kth phase and assuming that the switching at the slot boundaries is infinitely fast, the circuit behavior can be described by the following state equations:

$$\dot{X}_{n,k}(t) = A_k X_{n,k}(t) + B_k u(t), \; k = 1 \to N$$
$$Y_{n,k}(t) = C_k X_{n,k}(t) + D_k u(t), \; nT + \sigma_{k-1} < t < nT + \sigma_k \tag{1}$$

where

$$\sigma_k = \sum_{j=1}^{k} \tau_j$$

and τ_j is the width of the jth phase, with the understanding that $\sigma_0 = 0$ and $\sigma_N = T$. $X_{n,k}(t)$, $Y_{n,k}(t)$, and $u(t)$ are the state, output, and input signal vectors, respectively. A_k, B_k, C_k, and D_k are constant real matrices. The determination of A_k, B_k, C_k, and D_k matrices for a given circuit configuration can be carried out using the two-graph modified nodal analysis described in [2]. At the switching instants $t = nT + \sigma_k$, the state vectors are related by

$$X_{n,k}(\sigma_k) = F_k X_{n,k}(\sigma_k)^- + W_k \mu(nT + \sigma_k)$$
$$X_{(n+1),0}(\sigma_0)^+ = X_{n,N}(\sigma_0)^+, \quad F_0 = F_N \tag{2}$$