

Formation of a Fractal Basin Boundary in a Forced Oscillator

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Abstract—How does a fractal basin boundary arise in the sinusoidally forced Duffing's equation? We describe how the backwards system flow deforms a local stable manifold into the fractal boundary. Parts of the boundary are labeled in a way related to their time of formation. The truncated fractal boundary produced by a burst of sinusoidal forcing is briefly considered. The approach supplements the insights provided by the usual Poincaré map techniques.

I. INTRODUCTION

DUFFING'S equation is

$$\begin{aligned}\dot{\theta} &= v \\ \dot{v} &= \theta - \theta^3 - \delta v, \quad (\theta, v) \in \mathbf{R}^2.\end{aligned}\quad (1)$$

It has a hyperbolic equilibrium at $(0,0)$ and stable equilibria at $(\pm 1,0)$. The basin of attraction of each of the stable equilibria is defined to be the set of all initial conditions which tend to that equilibrium as time tends to infinity. The two basins are two-dimensional open sets and are separated by their common boundary, which is $W^s(0,0)$, the stable manifold of $(0,0)$ (see [1, fig. 2.2.3] or [2, fig. 1] or [3]). $W^s(0,0)$ is defined to be the set of initial conditions which tend to $(0,0)$ as time tends to infinity. The practical importance of the basin boundary $W^s(0,0)$ is that the ultimate fate of each initial condition is known once the position of $W^s(0,0)$ in phase space has been specified.

If small amplitude sinusoidal forcing is added to system 1 the equilibria become periodic orbits of small amplitude. $(0,0)$ becomes a hyperbolic periodic orbit γ and $(\pm 1,0)$ become attracting periodic orbits γ_{\pm} . $W^s(\gamma)$, the stable manifold of γ , and the basins of attraction of γ_{\pm} are defined as the set of initial conditions tending to γ or γ_{\pm} respectively. The basin boundary is the closure of $W^s(\gamma)$ and it separates the basin of γ_{+} from the basin of γ_{-} . Specifying the position of $W^s(\gamma)$ determines which attractor, γ_{+} or γ_{-} , an initial condition tends to. However, for certain parameter values, $W^s(\gamma)$ has a complicated and

fractal structure [2, fig. 2]. The arbitrarily fine structure of the fractal boundary obstructs prediction of which attractor a given initial condition tends to [3].

How can a fractal boundary arise from such relatively simple, smooth equations? This question is usually answered by reducing the system to a Poincaré map whose basin boundary can be studied using embeddings of horse-shoes [4], [5] or accessible saddle orbits [5], [6]. We describe the basin boundary from an alternative point of view and emphasize the temporal evolution of its fractal structure. This alternative approach should be regarded as supplementary to the usual approach; its main purpose is to provide additional insight into the fractal boundary.

For many engineering systems, forcing the equations with a constant amplitude sinusoid for all time is not realistic. For example, the forcing amplitude might be damped [7] or there may be a burst of forcing which lasts only a finite time. In the case of a burst of forcing, initial conditions eventually tend to one of the attractors for system 1 since the forcing ceases after a finite time. The basin boundary is not truly fractal, but a truncated fractal which has fractal self-similarity under several magnifications but "simple" structure at all greater magnifications. However, for practical purposes, even a truncated fractal may obstruct prediction of the ultimate fate of initial conditions. Since the truncated fractal approximates the true fractal boundary more closely as the burst time increases, one motivation for studying the case of forcing for all time is that it is the idealized limit of a system forced for a finite time.

After briefly introducing the forced Duffing's equation and sketching the usual approach to this problem, the flow in a sizeable region of phase space is linearized and part of the flow is reduced to a two-dimensional diffeomorphism on a surface of section within the linear region. The process by which this diffeomorphism deforms the local stable manifold of γ into a fractal boundary is described. A labeling scheme for parts of the boundary is presented and related to the time at which each part is formed. We suggest a way in which the theory for constant sinusoidal forcing might be applied to the case of a burst of forcing. The paper concludes by comparing the approach with the usual Poincaré map approach and indicating that the approach easily extends to attractors and the sinusoidally forced pendulum.

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II. SYSTEM DESCRIPTION

This section summarizes some basic facts about the forced Duffing's equation and mentions how the associated Poincaré map is usually studied to understand the nature of the basin boundary. A complete treatment of Duffing's equation and references to the literature may be found in [1].

When suspended in time, the sinusoidally forced Duffing's equation is

$$\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= \theta - \theta^3 - \delta v + g \cos \omega t \\ i &= 1, \quad (\theta, v, t) \in \mathbb{R}^2 \times \mathbb{S}^1. \end{aligned} \tag{2}$$

The phase space is cylindrical with period $T = 2\pi/\omega$ in the time direction. An equivalent formulation which is also useful in this paper is to "unwrap" the circular direction of the phase space \mathbb{S}^1 so that the unwrapped phase space is $\mathbb{R}^2 \times \mathbb{R}$ and the vector field is periodic in the time direction. The plane of constant time t is denoted by Σ_t . Since $\theta - \theta^3$ is an odd function of θ , system 2 has an important symmetry. Define a map σ from the phase space to itself by $\sigma(\theta, v, t) = (-\theta, -v, t + T/2)$. Then $(\dot{\theta}, \dot{v})$ evaluated at $\sigma(\theta, v, t)$ is the negative of $(\dot{\theta}, \dot{v})$ evaluated at (θ, v, t) . If the forcing amplitude g is small, system 2 has a hyperbolic periodic orbit γ near $(0,0) \times \mathbb{S}^1$ with Floquet multipliers λ, μ satisfying $-\mu > \lambda > 0$. γ is symmetric so that $(\theta, v, t) \in \gamma$ implies that $\sigma(\theta, v, t) \in \gamma$. The basins of attraction of the periodic orbits γ_{\pm} are separated by a basin boundary which is the closure of $W^s(\gamma)$. (There may be other attractors as well.) $W^s(\gamma)$ and $W^u(\gamma)$ are immersed two-dimensional submanifolds of the phase space. Computer simulation in [2] gives evidence that simple periodic attractors and fractal basin boundaries can occur simultaneously for forcing amplitudes just above those required for transversal intersection of $W^s(\gamma)$ and $W^u(\gamma)$.

The standard way of studying the immersion of $W^s(\gamma)$ uses Poincaré maps. The Poincaré map $P: \Sigma_0 \rightarrow \Sigma_0$ is the discrete map defined by flowing along trajectories of system 2 for time T . $\gamma \cap \Sigma_0$ is a hyperbolic fixed point of P and its stable manifold (with respect to P) is identical to $W^s(\gamma) \cap \Sigma_0$. Any stable manifold may be generated by backward integrating its local stable manifold, whose position is approximately known by linear theory. Using this method to sketch $W^s(\gamma \cap \Sigma_0)$ or calculate $W^s(\gamma \cap \Sigma_0)$ by computer provides useful insight into how $W^s(\gamma)$ is immersed in phase space. If $W^s(\gamma)$ intersects $W^u(\gamma)$ away from γ the picture is complicated and the labor of drawing or calculation limits the amount of $W^s(\gamma \cap \Sigma_0)$ that can be produced. Enough of the manifold may be generated to show the features and complexity of the situation but it is hard to make sense of all of the complexity.

One approach finds a rectangular region in Σ_0 on which an iterate of P acts like Smale's horseshoe map. Then it can be shown that a cross section of $W^s(\gamma \cap \Sigma_0)$ on the rectangular region is a Cantor set and most of the cross section can be labeled using symbolic dynamics (e.g., [4]).

III. CONSTRUCTING A SADDLE BOX

One natural way to model system 2 is to approximate it near γ by the forced linear system

$$\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= \theta - \delta v + g \cos \omega t \\ i &= 1 \end{aligned} \tag{3}$$

and then to choose a rotating coordinate system (x, y, t) which incorporates the effect of the forcing. It is also convenient to choose the x, y coordinates so as to diagonalize the system:

$$\begin{aligned} \dot{x} &= \lambda x \\ \dot{y} &= \mu y \\ i &= 1. \end{aligned} \tag{4}$$

The linear vector field 4 is assumed in the volume $\{(x, y, t) \mid |x| \leq a, |y| \leq a, t \in \mathbb{S}^1\}$ which is referred to here as a "saddle box" of size $2a$. Outside of the saddle box the system acts to reinject into the saddle box many of the trajectories which leave the saddle box.

This view of the system as a linear flow in a saddle box with reinjection is not an approximation; it may be realized by a change of coordinates as follows. Suppose system 2 has a hyperbolic periodic orbit γ symmetric under σ with Floquet exponents λ, μ . (Note that the Floquet exponents λ, μ are close to, but not identical to the eigenvalues λ, μ obtained by diagonalising (3).) Let $\theta_m < 1/\sqrt{3}$ be the maximum θ coordinate of points on γ and let the volume $V = \{(\theta, v, t) \mid |\theta| < \theta_0\}$ where $\theta_0 < (-\theta_m + \sqrt{4 - 3\theta_m^2})/2$. (Note that θ_m is $O(g)$ so that $\theta_0 = 1 - O(g)$.) Then there is a diffeomorphic coordinate change for V under which the vector field on V is given by system 4. In particular, a saddle box may be constructed in V with respect to the new coordinates; the size of the saddle box is only limited by the saddle box not overlapping the edges of V . The coordinate change is constructed in [8]; it is based on work by Hartman on local C^1 linearization of two-dimensional diffeomorphisms without the assumption of nonresonant eigenvalues [9].

The coordinate change described above "straightens out" γ so that it becomes $(0,0) \times \mathbb{S}^1$ and the local stable and unstable manifolds of γ become the planes $x = 0$ and $y = 0$, respectively, inside the saddle box. The lines $l^s = \{(0, a, t) \mid t \in \mathbb{S}^1\}$ and $l^u = \{(a, 0, t) \mid t \in \mathbb{S}^1\}$ are contained in $W^s(\gamma)$ and $W^u(\gamma)$, respectively. The cylindrical nature of the phase space causes Σ_0 and Σ_T to be identified so that l^s and l^u are topological circles. The two-dimensional half sides of the saddle box are labeled as shown in Fig. 1 with their labels positioned near their respective top edges. The saddle box sides S^s and S^u are defined by $S^s = S^s_- \cup l^s \cup S^s_+$ and $S^u = S^u_+ \cup l^u \cup S^u_-$. Much of the deformation of $W^s(\gamma)$ required for $W^s(\gamma)$ to be fractally immersed in the phase space occurs as it passes near γ ; linearizing the flow around γ in a saddle box allows this part of the deformation to be easily understood and calculated. One disadvantage is that the linearizing coordinate change is

example in Fig. 4(c). In this case, $l_{m\alpha}$ and $l_{\bar{m}\alpha}$ refer to more than one leaf each. Also not all possible multi-indexes have leaves associated with them. For example, the first stage leaves in Figs. 2 and 3 are $\{l_2, l_3, \dots, l_{\bar{2}}, l_{\bar{3}}, \dots, l_{-2}, l_{-3}, \dots, l_{-\bar{2}}, l_{-\bar{3}}, \dots\}$; first stage leaves with indexes smaller than 2 do not exist. However, there is a large subset of leaves for which these labeling problems are avoided. Define $A_K = \{\text{multi-indexes } \alpha \mid \text{if } m \text{ is an index in } \alpha \text{ then } |m| > K\}$. Then for some suitably large K , $\Lambda_K = \{l_\alpha \mid \alpha \in A_K\}$ contains leaves which, at each stage of their formation, consist of "tails" of exponentials C^1 close to l^s . If $|m| > K$ and $\alpha \in A_K$ then $\Gamma^{-1}(l_\alpha) \cap S_m^s$ has exactly two components as shown in Fig. 4a; these are labeled $l_{m\alpha}$ and $l_{\bar{m}\alpha}$. Moreover l_α exists for each $\alpha \in A_K$. Λ_K is contained in a narrow strip around l^s ; however, the narrow strip may contain leaves not in Λ_K . For example, if l_β is any leaf intersecting c_* then $l_{m\beta}$ is contained in the narrow strip for sufficiently large m .

The fractal structure arbitrarily close to l^s is reproduced around each leaf l_α of Λ . For if l_α was produced at stage n , then $\Gamma^n(l_\alpha)$ contains a segment of l^s and Γ^{-n} of a neighborhood of that segment of l^s contains l_α . Since Γ^{-n} is a diffeomorphism, the structure in that neighborhood of l^s is reproduced around l_α . For example, $l_m \rightarrow l^s$ as $|m| \rightarrow \infty$ implies that $l_{m\alpha} \rightarrow l_\alpha$ as $|m| \rightarrow \infty$. Thus each leaf of Λ is the limit of sequences of other leaves from both sides.

Λ is not the entire boundary since the boundary is a closed set and Λ is a proper subset of its closure $\bar{\Lambda}$. The Poincaré map and embedded horseshoe analysis shows that $\bar{\Lambda} - \Lambda$ is a complicated set which includes periodic orbits and their stable manifolds [1]. It is suggestive to think of $\bar{\Lambda} - \Lambda$ as being the union of leaves having multi-indexes of infinite length. Such leaves would remain arbitrarily close to Λ under iteration by Γ but are not on $W^s(\gamma)$ and never map to l^s .

VII. FRACTAL GENERATING REGIONS

Each stage in the construction of Λ may produce points in S^s which are lost from the subsequent stages of the construction because they lie outside $H - c_*$, the domain of Γ^{-1} . This effect may be controlled in the case of small forcing and damping by defining a fractal generating region G as explained below.

When backward integrating the local stable manifold of γ to obtain $W^s(\gamma)$, the overall effect of the system damping is that the manifold is repelled from the (forward time) attractors γ_+ and γ_- and attracted to infinity. Thus $W^s(\gamma) \cap S^s$ is formed starting with l^s and has an average tendency as it is formed to drift onto S_-^s and then to slip off the saddle box corner edge $S_-^s \cap S_-^u$. It is desirable to be able to define a region $G \subset H$ so that Γ^{-1} is defined on $G - c_*$ and in which all the fractal generation takes place. Since the only mechanism generating the fractal is the splitting by Γ^{-1} of leaves which cross c_* , G must contain c_* . Moreover, to ensure that Γ^{-1} acting on G captures all the fractal generation, G has the property that if $x \in G$ and $\Gamma^{-1}(x) \notin G$ then the backward trajectory through

$\Gamma^{-1}(x)$ cannot re-enter G . This ensures that portions of leaves which are mapped outside of G by Γ^{-1} never intersect c_* under further backward integration and so are not further split. Appendix B shows how subharmonic Melnikov theory guarantees the existence of G for sufficiently small forcing and damping.

For larger values of forcing and damping Γ^{-1} may not fully capture all the fractal structure because parts of the boundary outside the domain of Γ^{-1} may be reinjected to cross c_* and further complicate the fractal structure. Thus rigorously accounting for all the fractal structure by saddle box methods seems, unfortunately, to be limited to the case of small forcing and damping. However the approach still works for larger forcing and damping if the requirement that all the fractal structure be accounted for is relaxed. Since c_* is assumed to cross l^s twice, the backward trajectories through $c_* \cap l^s$ intersect S^u on l^u . The domain of R^{-1} , H can then be chosen as some neighborhood of the intersections $c_* \cap l^s$ such as two small discs covering the respective intersections. Γ^{-1} is defined on $H - c_*$ and any leaf l_α which intersects H will have an image $\Gamma^{-1}(l_\alpha \cap H)$ for which $l_{n\alpha}$ and $l_{\bar{n}\alpha}$ intersect H for n, \bar{n} larger than some sufficiently large K . Thus Γ^{-1} acting on H accounts for the subset Λ_K of the fractal structure.

VIII. TEMPORAL EVOLUTION OF THE FRACTAL

In applications it is important to note that the fine structure of a fractal boundary takes a long time to evolve by the process of backward integrating the local stable manifold. Indeed, the notion that the infinitely fine fractal structure takes an infinite time to evolve helps to make the occurrence of fractal basin boundaries in models of physical phenomena less strange. Thus in determining the basin boundary of system 2 by forward integrating initial conditions for some finite time, an initial condition away from the fine structure will converge near to one of the attractors while an initial condition amongst the fine structure will not converge.

The time evolution of the fractal may be quantified since the approximate evolution time of a leaf is simply related to its multi-index. It follows from the definition of the leaf labeling that $l_{m\beta}$ is formed at a time approximately mT after the formation of l_β . If l^s is considered to be formed at time zero, a leaf $l_{m_k m_{k-1} \dots m_1}$ is formed at approximately $(m_1 + m_2 + \dots + m_k)T$. Thus the leaves formed in time L are roughly $\{l_{m_k m_{k-1} \dots m_1} \mid m_1 + m_2 + \dots + m_k < LT^{-1}\}$.

The following paragraph speculates how the saddle box theory presented above might be applied to the case of a burst of constant amplitude forcing of finite duration. The scaling of the precision needed to determine the basin boundary is roughly estimated.

The system is assumed to have three regimes; no forcing for negative time, a burst of constant amplitude sinusoidal forcing from time 0 to time L , and no forcing after time L . The forced regime assumptions are the same as those for the previous sections. The system is considered to be a

flow suspended in time even when it is unforced. Since the system is unforced after time L , it has exactly two attractors at $(\pm 1, 0) \times \{t > L\}$, each with its own basin of attraction. The basin boundary is the stable manifold of the hyperbolic orbit $(0, 0) \times \{t > L\}$. The objective is to estimate the fineness of the basin boundary at time 0 (i.e., the basin boundary $\cap \Sigma_0$). The basin boundary at time greater than L is just $W^s(0, 0)$ for the unforced system and it intersects the saddle box side S_-^s as a series of straight lines parallel to and including l^s in the saddle box coordinates appropriate to the unforced system. The basin boundary at time 0 is the basin boundary at time L integrated backwards for time L through the forced regime. If the coordinates are changed to the saddle box coordinates for the forced regime, the boundary at time L becomes a series of roughly sinusoidal curves on S^s . The construction of the boundary proceeds just as in the previous analysis except that the initial curve is not l^s , but the series of curves. However, if $W^s(\gamma)$ for system 2 is assumed to be an attractor for the backward flow in the forced regime then any curve close to l^s will asymptotically tend to $W^s(\gamma)$ for system 2. Thus the fineness of the boundary at time 0 may be estimated by the fineness of the boundary produced by backwards integrating l^s for time L . The leaves produced are estimated to be $\{l_{m_k m_{k-1} \dots m_1} | m_1 + m_2 + \dots + m_k < LT^{-1}\}$. Both the boundary at time 0 and its approximation for the purpose of estimating fineness are truncated fractals. The finest leaf structure arises from the indices with the highest absolute value of m , so that the leaves representative of the finest structure are $l_{\pm m}$ and $l_{\pm \bar{m}}$ where $m \approx LT^{-1}$. For large L , l_m is part of an exponential tail $\{(e^{\lambda t}, t) | -m \leq t + r \leq 1 - m\}$ (Here r is the reinjection time, t is the time inside the saddle box and $t + r$ is the total time of formation of l_m from l^s .) The x coordinate of l_m is approximately $e^{\lambda(-|m|-r)} \approx e^{-\lambda LT^{-1}} e^{-\lambda r}$. Thus the finest structure of the truncated fractal consists of exponential segments such as l_m whose x coordinates scale like $e^{-\lambda LT^{-1}}$. Therefore the minimum distance between $l_{\pm m}$, $l_{\pm \bar{m}}$ and l^s and hence the measurement precision required to distinguish all parts of the boundary scale like $e^{-\lambda LT^{-1}}$. If the measurement precision or system noise length scale exceeds the minimum distance, then deterministic prediction at time 0 of which attractor the system will tend to must fail in at least part of the phase space.

IX. CONCLUSIONS

Insight into the fractal immersion of $W^s(\gamma)$ may be gained by studying its intersection with the saddle box sides as it evolves from the local stable manifold. Fractal formation requires the transversal intersection of $W^s(\gamma)$ and $W^u(\gamma)$ and occurs for a sufficiently large forcing amplitude to damping ratio (this conclusion is well known). The distortion of surfaces required to form the fractal may be understood by considering the effect of a backwards linear flow in a saddle box on curves on the saddle box side S^u . For small forcing and damping the two-dimen-

sional map Γ^{-1} completely captures the fractal formation. It is clear that fractal formation occurs at higher values of forcing and damping, but not all the features of the boundary are accounted for by Γ^{-1} . Parts of $W^s(\gamma)$ intersecting the saddle box side S^s may be labeled to indicate how they were formed from the local stable manifold and their approximate time of formation. The importance in applications of considering when the various parts of the boundary are formed is emphasized.

This paper only considers the basin boundary of the forced Duffing's equation. However the saddle box method applies with equal force to other important examples. The method applies to the sinusoidally forced, damped pendulum for small forcing and damping. (The suspended phase space is $S^1 \times R \times S^1$ and the reinjection to the saddle box occurs via the periodicity in the pendulum angle.) By integrating the local unstable manifold forward in time to study the intersection of $W^u(\gamma)$ with S^u , the (presumed) strange attractor $\overline{W^u(\gamma)}$ of the forced Duffing's equation and the forced pendulum can also be studied using the saddle box method.

The Poincaré map and saddle box approaches are similar but are suggestive of different insights. They are similar in that the Poincaré map is a diffeomorphism on Σ_0 whereas Γ^{-1} is a diffeomorphism on the saddle box side S^s . Indeed, the system flow maps $W^s(\gamma) \cap S^s$ into $W^s(\gamma) \cap \Sigma_0$. More work is required to define Γ^{-1} than the Poincaré map. The saddle box approach exploits the system symmetry σ to simplify the formation of $W^s(\gamma)$. The two approaches treat time differently. The Poincaré map advances time by the system period T every iteration while each iteration of Γ^{-1} corresponds to decreasing time by an amount which varies to infinity. It is natural in the saddle box approach to partition $W^s(\gamma) \cap S^s$ with Σ_0 and label the leaves accordingly; this differs from the natural partitions of $W^s(\gamma)$ in the Poincaré map approach. Thus the saddle box approach encourages new ways of labeling and organizing parts of the basin boundary. In future work, the author hopes to use the leaves on the saddle box side to reduce the generation of the basin boundary to a one-dimensional map in a natural way.

APPENDIX A

c_* IS APPROXIMATELY SINUSOIDAL

Choose a point q on the homoclinic orbit of the unforced, undamped system so that the plane Q normal to the vector field at q is close to the saddle box side S^s of the forced and damped system. When small forcing and damping of order ϵ are turned on, Melnikov theory [1] shows that $W^u(\gamma) \cap Q$ is a sinusoidal curve to order ϵ . "Projecting" $W^u(\gamma) \cap Q$ onto S^s along the system flow to obtain $c_* = W^u(\gamma) \cap S^s$ will not alter its sinusoidal character since S^s is close to Q . Changing coordinates according to Section III in order to "straighten out" S^s will also preserve the sinusoidal character of c_* since the coordinate change is approximately the rotating coordinate change transforming system 3 into system 4.

APPENDIX B

APPLICATION OF SUBHARMONIC MELNIKOV THEORY

First consider the unforced, undamped system; this has a periodic orbit η of period $(m + (1/2))T$ inside the homoclinic loop which contains the saddle box corner edge $S_+^s \cap S_+^u$ for some integer m . (The saddle box supplied by the theorem in [8] is large enough to allow a small value of m to be chosen.) When positive forcing and damping of order ϵ are turned on, η is no longer a periodic orbit (only orbits of period mT are preserved) and subharmonic Melnikov theory [1] shows that for small enough ϵ , $P^{2m+1}(\eta)$ lies inside η to order ϵ . Integrating η forward for a time $(2m+1)T$ forms a cylinder in $\mathbf{R}^2 \times \mathbf{R}$ whose intersection with any time slice Σ_t is close to η . The volume V inside this cylinder and all its $(2m+1)T$ time translates in $\mathbf{R}^2 \times \mathbf{R}$ is then an invariant volume only intersecting the saddle box in S_+^s and S_+^u and containing $S_+^s \cap S_+^u$. By similarly applying subharmonic Melnikov theory to an orbit $\hat{\eta}$ outside the homoclinic loop of period $2\hat{m}$ (only odd subharmonics are preserved outside the homoclinic loop), we obtain a volume \hat{V} which is backward invariant under the flow and which intersects the saddle box only in the sides $S_-^s, S_-^u, S_+^s, S_+^u$ and contains the saddle box corner edges $S_-^s \cap S_-^u$, and $S_+^s \cap S_+^u$. Backward trajectories starting in $G' = S^s - ((V \cup \hat{V}) \cap S^s)$ cannot enter V since V is invariant and, if they enter \hat{V} , cannot leave \hat{V} since \hat{V} is backward invariant. Define G to be $G' \cap H$. Suppose $x \in G$ and $\Gamma^{-1}(x) \notin G$. If $\Gamma^{-1}(x) \notin G'$ then $\Gamma^{-1}(x) \in \hat{V}$. If $\Gamma^{-1}(x) \notin H$ then the backward trajectory through $\Gamma^{-1}(x)$ does not intersect S^u and so must intersect \hat{V} because of the way in which \hat{V} encloses the saddle box corner edges. In either case the backward trajectory through $\Gamma^{-1}(x)$ remains in \hat{V} and cannot re-enter G .

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REFERENCES

- [1] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. New York, Springer-Verlag, 1983.
- [2] F. C. Moon and G-X. Li, "Fractal basin boundaries and homoclinic orbits for periodic motion in a two well potential," *Phys. Rev. Lett.*, vol. 55, no. 14, pp. 1439-1442, Sept. 1985.
- [3] S. W. McDonald, G. Grebogi, E. Ott, and J. A. Yorke, "Final state sensitivity: an obstruction to predictability," *Phys. Lett. A*, vol. 99, no. 9, pp. 415-418, 1983.
- [4] K. Hockett and P. Holmes, "Josephson's junction, annulus maps, Birkhoff attractors, horseshoes and rotation sets," *Ergod. Theory Dyn. Syst.*, vol. 6, pp. 205-239, 1986.
- [5] K. T. Alligood, L. Tedeschini-Lalli, and J. A. Yorke, "Metamorphoses: Sudden jumps in basin boundaries," preprint, George Mason Univ., Fairfax, VA, Dec. 1985.
- [6] C. Grebogi, E. Ott, and J. A. Yorke, "Basin boundaries metamorphoses: changes in accessible boundary orbits," *Physica 24D*, pp. 243-262, Jan./Feb. 1987.
- [7] M. Varghese and J. S. Thorp, "Truncated fractal basin boundaries in forced pendulum systems," *Phys. Rev. Lett.*, vol. 60, no. 8, pp. 665-668, Feb. 1988.
- [8] I. Dobson, "Describing complicated basin boundaries in a forced oscillator," chap. 3, Ph. D. dissertation, Cornell Univ., Ithaca, NY, 1989.
- [9] P. Hartman, "On local homeomorphisms of Euclidean space," *Soc. Matematica Mexicana Boletin*, vol. 5, pp. 220-241, Oct. 1960.

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