Counterexamples to a Method for Identifying Hopf Bifurcations Without Eigenvalue Calculation

Jaime Arroyo, *Student Member, IEEE*, and Ian Dobson, *Fellow, IEEE*

***Abstract*—We give counterexamples to the numerical method for detecting Hopf bifurcations in electric power systems by singularity of a symmetrized Jacobian in Y. Zhou, V. Ajjarapu, A fast algorithm for identification and tracing of voltage and oscillatory stability margin boundaries, IEEE Proceedings, vol. 93, no. 5, May 2005, pp. 934–946. The counterexamples include some simple matrix examples and a single machine infinite bus power system.**

***Index Terms*—Bifurcation, numerical analysis, power system dynamic stability.**

I. Detecting Electric Power System Oscillations

THERE is great economic incentive to be able to operate bulk electric power systems near but not beyond their stability limits such as the oscillatory stability limit associated with Hopf bifurcation [1]–[4]. Oscillations in power systems need to be avoided because they can damage equipment, interfere with system controls and be a factor in complicated cascading blackouts [5]. Hopf bifurcations can be detected by eigenanalysis of differential-algebraic power system models, but it remains worthwhile to seek to improve the efficiency and accuracy of these numerical methods.

II. Method of Identifying Hopf Bifurcations in [6]

A standard differential-algebraic power system model has the form

$$\begin{aligned} \dot{x} &= f(x,y) \\ 0 &= g(x,y). \end{aligned} \tag{1}$$

Define the Jacobian matrices

$$A_{\text{total}} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \tag{2}$$

$$A_{\text{sys}} = f_x - f_y g_y^{-1} g_x. \tag{3}$$

Let λ_1 be the maximum eigenvalue of the symmetric matrix $A_{\text{total}} + A_{\text{total}}^T$. Then Zhou and Ajjarapu [6, section IVC] make the following claims.

Claim 1: Hopf bifurcation of (1) implies $\lambda_1 = 0$.

Claim 2: $\lambda_1 \leq 0$ for normal power system operation.

Claim 3: $\lambda_1 = 0$ indicates a Hopf bifurcation.

Manuscript received July 18, 2006; revised November 13, 2006. Funding in part from CONACYT Mexico with scholarship number 143893 is gratefully acknowledged. This paper was recommended by Associate Editor M. di Bernardo.

J. Arroyo was a visiting scholar in the ECE Department, University of Wisconsin, Madison, WI 53706 USA. He is also with the CINVESTAV, Mexico (e-mail: jarroyo@gdl.cinvestav.mx).

I. Dobson is with the ECE Department, University of Wisconsin, Madison, WI 53706 USA (e-mail: dobson@engr.wisc.edu).

Digital Object Identifier 10.1109/TCSII.2006.890404

The standard eigenvalue condition implied by Hopf bifurcation of (1) and used to detect Hopf bifurcations numerically is that A_{sys} has eigenvalues $\pm j\omega$ with $\omega \neq 0$.

III. Matrix Counterexamples

Suppose that the system is two-dimensional with linearization

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = A_{\text{sys}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{4}$$

In the case (4) of no algebraic equations, A_{sys} and A_{total} coincide

$$A_{\text{total}} + A_{\text{total}}^T = A_{\text{sys}} + A_{\text{sys}}^T = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix} \tag{5}$$

and the maximum eigenvalue of $A_{\text{total}} + A_{\text{total}}^T$ is

$$\begin{aligned} \lambda_1 &= a+d+\sqrt{(a+d)^2+(b+c)^2-4ad} \\ &= \text{tr}(A_{\text{sys}})+\sqrt{(\text{tr}(A_{\text{sys}}))^2+(b-c)^2-4\det(A_{\text{sys}})}. \end{aligned} \tag{6}$$

Therefore, the condition $\lambda_1 = 0$ is equivalent to

$$4\det(A_{\text{sys}}) = (b-c)^2 \quad \text{and} \quad \text{tr}(A_{\text{sys}}) \leq 0. \tag{7}$$

On the other hand, the condition for Hopf bifurcation is that A_{sys} has nonzero eigenvalues $\pm j\omega$, or, equivalently, that

$$\text{tr}(A_{\text{sys}}) = 0 \quad \text{and} \quad \det(A_{\text{sys}}) > 0. \tag{8}$$

Conditions (7) and (8) are different and do not imply each other, contradicting claims 1 and 3. For example, consider the following values for the matrix A_{sys} in (4):

$$A_{\text{sys}} = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} \tag{9}$$

is at a Hopf bifurcation with eigenvalues $\pm j\sqrt{3}$ but $\lambda_1 = 2$, and

$$A_{\text{sys}} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \tag{10}$$

has $\lambda_1 = 0$ but has both eigenvalues -1 and is not at a Hopf bifurcation.

Consider (4) with the change of coordinates

$$\begin{aligned} x_1' &= x_1 + kx_2 \\ x_2' &= x_2. \end{aligned} \tag{11}$$

Then, in the new coordinates

$$\begin{aligned} A'_{\text{sys}} &= \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a+ck & b-(a-d)k-ck^2 \\ c & d-ck \end{pmatrix}. \end{aligned} \tag{12}$$

The eigenvalues, determinant, and trace of A_{sys} are preserved by the change of coordinates. Let λ_1' be the maximum eigenvalue of $A'_{\text{sys}} + A'^T_{\text{sys}}$. Then $b' - c' = b - c - (a-d)k - ck^2$ and (6) and the generic assumption that $c \neq 0$ imply that

$$\lambda_1' \to \infty \quad \text{as} \quad k \to \infty. \tag{13}$$

More generally, consider a state space with more than two dimensions and a linearization $\dot{x} = Ax$. The system is assumed to be at a Hopf bifurcation so that there is a single, simple pair of complex eigenvalues $\pm j\omega$. Then, by choosing a basis with the first two vectors in the eigenspace corresponding to $\pm j\omega$ and the remaining vectors in the generalized eigenspaces corresponding to all the other eigenvalues, there is a coordinate change in which A appears in the block matrix form

$$\begin{pmatrix} A_{\text{sys}} & 0 \\ 0 & M \end{pmatrix}. \tag{14}$$

We make the generic assumption that the entry c of A_{sys} is not zero. Then, a further coordinate change (11) is applied to the first two coordinates so that A_{sys} in (14) is replaced by A'_{sys} to obtain

$$A' = \begin{pmatrix} A'_{\text{sys}} & 0 \\ 0 & M \end{pmatrix}. \tag{15}$$

Then (13) shows that as k becomes arbitrarily large the maximum eigenvalue of $A' + A'^T$ is the maximum eigenvalue λ_1' of $A'_{\text{sys}} + A'^T_{\text{sys}}$ and becomes arbitrarily large. This shows how λ_1 can be arbitrarily large at a Hopf bifurcation. λ_1 depends on the choice of coordinates whereas the occurrence of Hopf bifurcation is coordinate independent.

For the case of no algebraic equations, claims 1 and 3 are correct if the coordinates are chosen so that the matrix A_{sys} is in Jordan real canonical form (in real canonical form, recall that simple complex eigenvalues $\sigma \pm j\omega$ correspond to matrix blocks $\begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix}$ along the diagonal). Unfortunately, the transformation to real canonical form requires knowledge of the eigenstructure.

To consider cases with algebraic equations, augment (4) with the algebraic state variable y and a linearized algebraic equation

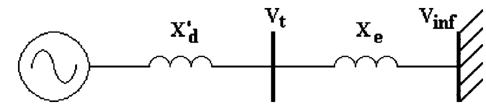

Fig. 1. Single machine infinite bus system.

$0 = gx_2 + hy.$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & g & h \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix}. \tag{16}$$

Then

$$A_{\text{total}} = \begin{pmatrix} A_{\text{sys}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & g \end{matrix} & h \end{pmatrix}$$

and

$$A_{\text{total}} + A_{\text{total}}^T = \begin{pmatrix} A_{\text{sys}} + A_{\text{sys}}^T & \begin{matrix} 0 \\ g \end{matrix} \\ \begin{matrix} 0 & g \end{matrix} & 2h \end{pmatrix}. \tag{17}$$

The case $g = 0$ and $h = -1$ yields the linearized algebraic equation $-y = 0$ and reduces to the case (4) without algebraic equations considered above. More generally, the dynamics of (16) are governed by A_{sys} and are independent of the constants g and h. In particular, if the system is at a Hopf bifurcation A_{sys} has eigenvalues $\pm j\omega$ for $\omega \neq 0$ regardless of the constants g and h. However, it is clear that λ_1 generally depends on g and h. For example, for small g and sufficiently large h, $\lambda_1 \approx 2h$, so that λ_1 tends to infinity as $h \to \infty$. These observations do not rely on the differential equations in (16) not including the algebraic variable y because

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & g \\ 0 & 0 & h \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \tag{18}$$

yields the same matrix (17) and the same observations.

We have not obtained additional conditions under which the results of [6] become valid. One aspect of the problem is the non-normal matrices that can occur in linearizations of power system dynamics. For example, non-normality is implied by the strong eigenvalue resonances that are conjectured to explain some power system oscillations [7]. Another aspect in cases with algebraic equations is directly relating the oscillatory system stability to simple properties of the matrices A_{total} or $A_{\text{total}} + A_{\text{total}}^T$. However, an anonymous reviewer correctly observed that the arguments in [6] do imply that $\lambda_1 < 0$ precludes Hopf bifurcation.

IV. Power-System Counterexample

Consider the single machine infinite bus system shown in Fig. 1. The generator is described by the standard flux-decay model with a single time constant exciter. The particular equations used are in [8, sec. 8.22]. The load flow solution is given by $V_t = 1.0\angle 15°$ and $V_{\text{inf}} = 1.05\angle 0°$. The generator parameters are $T'_{do} = 9.6$, $H = 3.2$, $D = 0.0$, $X_d = 2.5$, $X_q = 2.1$, $X'_d = 0.39$, $R_s = 0.0$. (All data is in per unit except that time constants are in seconds.) The line parameters are $R_e = 0.0$ and

TABLE I
EIGENVALUES AT A STABLE OPERATING POINT

matrix	eigenvalues	comment
A_{sys}	$-2.5885 \pm j8.5020$	stable
	$-0.0871 \pm j7.1139$	
$A_{sys} + A_{sys}^T$	-722.59	
	711.95	max eigenvalue> 0
	-52.263	
	52.191	
$A_{total} + A_{total}^T$	-2005.0	
	1995.0	$\lambda_1 > 0$
	-97.824	
	96.454	
	1.9852	
	-0.8773	
	0.6464	
	-0.5933	

TABLE II
EIGENVALUES AT HOPF BIFURCATION

matrix	eigenvalues	comment
A_{sys}	$-2.6756 \pm j10.466$	
	$0.0000 \pm j7.1790$	indicates Hopf
$A_{sys} + A_{sys}^T$	-1083.9	
	1073.3	max eigenvalue> 0
	-52.358	
	52.331	
$A_{total} + A_{total}^T$	-3019.2	
	3009.2	$\lambda_1 > 0$
	-97.824	
	96.454	
	1.9852	
	-0.8774	
	0.6465	
	-0.5932	

$X_e = 0.5$. The exciter parameters are $T_a = 0.2$ and the excitation system gain K_a. K_a is varied to yield a normal operating point and a Hopf bifurcation.

At $K_a = 400$ there is a normal, stable operating point and the eigenvalues in Table I show that $\lambda_1 > 0$, contradicting claim 2. At $K_a = 602.833$ there is a Hopf bifurcation and the eigenvalues in Table II show that $\lambda_1 > 0$, contradicting claim 1. Moreover, the maximum eigenvalue of $A_{sys} + A_{sys}^T$ is also positive.

V. CONCLUSION

The counterexamples show that the method proposed in [6] for detecting Hopf bifurcations without eigenvalue calculations is generally incorrect. The method fails to detect Hopf bifurcations in some simple power system and low dimensional examples. Moreover there are low dimensional examples for which the method incorrectly predicts a Hopf bifurcation. The method is not independent of the coordinate system chosen whereas eigenvalues and the occurrence of Hopf bifurcation are independent of the coordinate system. We note that Alvarado [9] has suggested applying a Kronecker sum approach to detecting Hopf bifurcations in power systems that bypasses eigenvalue calculation.

REFERENCES

[1] G. Rogers, *Power System Oscillations*. Norwell, MA: Kluwer, 2000.
[2] Analysis and control of power system oscillations, Cigré Task Force 07 of Advisory Group 01 of Study Committee 38, Paris, France, Dec. 1996.
[3] Eigenanalysis and frequency-domain methods for system dynamic performance, IEEE Power System Engineering Committee, IEEE Publ. 90TH0292-3-PWR, 1989.
[4] Inter-area oscillations in power systems, IEEE Power Engineering Society Systems Oscillations Working Group, IEEE Publ. 95 TP 101, Oct. 1994.
[5] V. Venkatasubramanian and Y. Li, "Analysis of 1996 Western American electric blackouts," in *Proc. Bulk Power Syst. Dynamics and Contr.-VI*, Cortina d'Ampezzo, Italy, Aug. 2004.
[6] Y. Zhou and V. Ajjarapu, "A fast algorithm for identification and tracing of voltage and oscillatory stability margin boundaries," *Proc. IEEE*, vol. 93, no. 5, pp. 934–946, May 2005.
[7] I. Dobson, J. Zhang, S. Greene, H. Engdahl, and P. W. Sauer, "Is strong modal resonance a precursor to power system oscillations?," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 48, no. 3, pp. 340–349, Mar. 2001.
[8] P. W. Sauer and M. A. Pai, *Power System Dynamics and Stability*. Englewood Cliffs, NJ: Prentice-Hall, 1998.
[9] F. L. Alvarado, "Bifurcations in nonlinear systems-computational issues," in *Proc. IEEE Int. Symp. Circuits Syst.*, New Orleans, LA, May 1990, vol. 2, pp. 922–925.